

PROPERTIES OF THE INTRINSIC FLAT DISTANCE

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ABSTRACT. Here we explore the properties of Intrinsic Flat convergence, proving a number of theorems relating it to Gromov-Hausdorff convergence and proving new Gromov-Hausdorff and Intrinsic Flat compactness theorems including the Tetrahedral Compactness Theorem. We introduce the sliced filling volume and related notions which are continuous under intrinsic flat convergence and can be applied to prevent the disappearance of points in the limit. An Arzela-Ascoli Theorem is proven as well.

This is a preliminary preprint covering everything presented at MIT on Feb 15, 2011 as well as additional material added through August 2012. Many more results still need to be texed. Everything stated here has been proven in complete detail here and may be applied by those who need the results.

1. INTRODUCTION

The Intrinsic Flat distance between Riemannian manifolds has been applied to study the stability of the Positive Mass Theorem, the rectifiability of Gromov-Hausdorff limits of Riemannian manifolds and smooth convergence away from singular sets. Here we present properties of Riemannian manifolds which are conserved under intrinsic flat convergence. The initial notion of the intrinsic flat distance and lower semicontinuity of mass and the continuity of filling volumes of balls is joint work with Stefan Wenger appearing in [19] and [20]. Here we provide more detailed proofs of the properties described in [19] and prove additional properties as well as new compactness theorems.

The intrinsic flat distance is defined using Gromov's idea of isometrically embedding two Riemannian manifolds into a common metric space. Rather than measuring the Hausdorff distance between the images as Gromov did when defining the Gromov-Hausdorff distance in [10], one views the images as integral currents in the sense of Ambrosio-Kirchheim in [1] and take the flat distance between them. Wenger's compactness theorem states that a sequence of Riemannian manifolds with a uniform upper bound on volume, boundary volume and diameter has a converging subsequence [22]. The limit spaces are countably \mathcal{H}^m rectifiable metric spaces called integral current spaces [20]. In Section 2, we review Gromov-Hausdorff convergence, intrinsic flat convergence and a few key theorems of Ambrosio-Kirchheim that were never reviewed and applied in [20]. We also review the most important results from [20] that will be applied here. Recall that under intrinsic flat convergence thin regions can disappear in the limit and so it is essential for us to understand what points disappear and which remain.

In Section 3 we explore the metric properties of integral current spaces. We describe the balls in the spaces, their isometric products, level sets of Lipschitz functions called slices, spheres, estimates on the masses of the slices and on the filling volumes of slices and spheres. We recall how in prior work of the author with Wenger in [19], the filling volumes of spheres were estimated using contractibility functions much in the style of [7] and [9]. This was applied in [19] to prevent the disappearance of points in the limit

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because the masses of balls in the limit would be bounded below by the filling volumes of spheres. Then we introduce the new notion of a *sliced filling volume* [Defn 3.20] and prove a new Gromov-Hausdorff compactness theorem [Theorem 3.23]. We then relate the sliced filling volumes to a new notion called the tetrahedral property [Defn 3.30] and the integral tetrahedral property [Defn 3.36]. This provides another Gromov-Hausdorff Compactness Theorem [Theorem 3.41]. Here we state the three dimensional version of this theorem:

Theorem 1.1. *Given $r_0 > 0, \beta \in (0, 1), C > 0, V_0 > 0$, If a sequence of Riemannian manifolds, M_i^3 , has $\text{Vol}(M_i^3) \leq V_0$, $\text{Diam}(M_i^3) \leq D_0$ and the C, β tetrahedral property for all balls, $B_p(r) \subset M_i^3$, of radius $r \leq r_0$:*

$$\exists p_1, p_2 \in \partial B_p(r) \text{ such that } \forall t_1, t_2 \in [(1 - \beta)r, (1 + \beta)r] \text{ we have}$$

$$\inf\{d(x, y) : x \neq y, x, y \in \partial B_p(r) \cap \partial B_{p_1}(t_1) \cap \partial B_{p_2}(t_2)\} \in [Cr, \infty)$$

then a subsequence of the M_i converges in the Gromov-Hausdorff sense. See Figure 1.

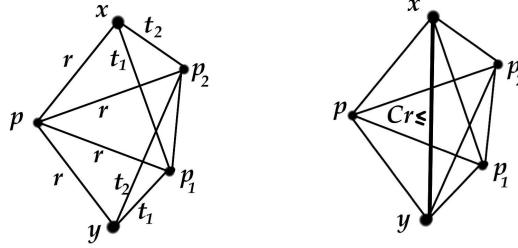


FIGURE 1. Tetrahedral Property

In fact this is a noncollapsing compactness theorem, since the volumes of balls are bounded below. Later in the paper we will see that these sequences also have intrinsic flat limits, the intrinsic flat limits agree with the Gromov-Hausdorff limits and thus the limits are countable \mathcal{H}^3 rectifiable metric spaces. Section 3 closes with the definition of interval filling volumes [Defn 3.43], sliced interval filling volumes [Defn 3.45] and corresponding estimates on mass based upon these notions. A short description of the key steps needed in the proof of the Tetrahedral Compactness Theorem appears in [17].

In Section 4 we begin to examine when sequences of points $p_i \in M_i$ disappear under intrinsic flat convergence, $M_i \xrightarrow{\mathcal{F}} M_\infty$. We define converging and Cauchy sequences of points $p_i \in M_j$ [Definitions 4.1 and 4.2]. We prove all points $p_\infty \in M_\infty$ are limits of converging sequences of points $p_i \in M_i$ and that the diameters are lower semicontinuous $\liminf_{i \rightarrow \infty} \text{Diam}(M_i) \geq \text{Diam}(M_\infty)$ [Theorems 4.3 and 4.5]. Then we prove Theorem 4.6, that M_∞ is the Gromov-Hausdorff limit of regions, N_i within the M_i . If $\text{Vol}(M_i) \rightarrow \text{Vol}(M_\infty)$ then $\text{Vol}(M_i \setminus N_i) \rightarrow 0$ [Remark 4.7].

In the next few subsections we prove the continuity of the sliced filling volumes [Theorem 4.20], the interval filling volumes [Theorem 4.23] and the sliced interval filling volumes [Theorem 4.24]. These theorems are built upon the fact that Wenger and the author had shown the continuity of the filling volume in [19] (reproven here in more detail as Theorem 2.46). We first prove that spheres converge as well as slices of spheres and then obtain strong enough estimates on the fillings of the slices of the spheres to prove our theorems.

We apply the sliced filling volumes to prevent the disappearance of Cauchy sequences of points [Theorem 4.27]. This leads to a Bolzano-Weierstrass Theorem (not yet texed) and a few Arzela-Ascoli Theorems (the first of which has been texed as Theorem 4.29). We also improve the compactness theorems mentioned earlier stating that the Gromov-Hausdorff and Intrinsic Flat limit spaces agree and are thus countable \mathcal{H}^m rectifiable spaces. We have a (integral) sliced filling compactness theorem [Theorem 4.30] and a tetrahedral compactness theorem [Theorem 4.31]. This whole section could also be redone with theorems corresponding to interval fillings and sliced interval fillings (although this has not yet been texed).

One may recall Wenger's Compactness Theorem: if a sequence of M_i has a uniform upper bound of diameter, volume and boundary volume, then a subsequence converges in the intrinsic flat sense to an integral current space possibly the zero space [22]. One contrast between Wenger's Compactness Theorem and these new compactness theorems, is that our limit spaces are never the zero space and so naturally are subjected to far stronger hypotheses. It should be noted that Wenger's Compactness Theorem is never applied to prove anything in this paper. Thus, those who might wish to find a new proof for his theorem, could use material found here. Naturally any proof of his compactness theorem would have to allow for sequences with disappearing points.

This paper is still in progress. In addition to inserting the promised Bolzano-Weierstrass Theorem and additional Arzela-Ascoli Theorems, there will be sections on the consequences of continuity and additional slicing convergence theorems. As the author is working with other mathematicians on applications of the intrinsic flat convergence to questions arising in General Relativity, new properties may also be seen to be necessary and they will be proven here. The goal is to include all theorems which require a deep understanding of the work of Ambrosio-Kirchheim in this single paper so that they may be verified and refereed by experts in that area. Then the applications to General Relativity can be explored without requiring everyone to master geometric measure theory. Every theorem stated in this preprint has been proven in detail and the proofs are provided here.

While it is not necessary, students reading this paper are encouraged to read Burago-Burago-Ivanov's textbook [2] which provides a thorough background in Gromov-Hausdorff convergence and also to read the authors joint paper with Wenger [20]. Those who would like to understand the Geometric Measure Theory more deeply should read Morgan's textbook [16] or Fanghua Lin's textbook [15] and then the work of Ambrosio-Kirchheim [1]. We review all theorems applied in this paper in the review section.

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2. REVIEW

In this section we review the Gromov-Hausdorff distance introduced by Gromov in [10], then various topics from Ambrosio-Kirchheim's work in [1], then intrinsic flat convergence and integral current spaces from prior work of the author with Wenger in [20] and end with a review of filling volumes which are related to Gromov's notion from [9] but defined using the work of Ambrosio-Kirchheim.

2.1. Review of the Gromov-Hausdorff Distance. First recall that $\varphi : X \rightarrow Y$ is an isometric embedding iff

$$(1) \quad d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \quad \forall x_1, x_2 \in X.$$

This is referred to as a metric isometric embedding in [13] and it should be distinguished from a Riemannian isometric embedding.

Definition 2.1 (Gromov). *The Gromov-Hausdorff distance between two compact metric spaces (X, d_X) and (Y, d_Y) is defined as*

$$(2) \quad d_{GH}(X, Y) := \inf d_H^Z(\varphi(X), \psi(Y))$$

where Z is a complete metric space, and $\varphi : X \rightarrow Z$ and $\psi : Y \rightarrow Z$ are isometric embeddings and where the Hausdorff distance in Z is defined as

$$(3) \quad d_H^Z(A, B) = \inf\{\epsilon > 0 : A \subset T_\epsilon(B) \text{ and } B \subset T_\epsilon(A)\}.$$

Gromov proved that this is indeed a distance on compact metric spaces: $d_{GH}(X, Y) = 0$ iff there is an isometry between X and Y . When studying metric spaces which are only precompact, one may take their metric completions before studying the Gromov-Hausdorff distance between them.

Definition 2.2. *A collection of metric spaces is said to be equibounded or uniformly bounded if there is a uniform upper bound on the diameter of the spaces.*

Remark 2.3. *We will write $N(X, r)$ to denote the number of disjoint balls of radius r in a space X . Note that X can always be covered by $N(X, r)$ balls of radius $2r$.*

Note that Ilmanen's Example of [20] of a sequence of spheres with increasingly many splines is not equicompact, as the number of balls centered on the tips approaches infinity.

Definition 2.4. *A collection of spaces is said to be equicompact or uniformly compact if they have a common upper bound $N(r)$ such that $N(X, r) \leq N(r)$ for all spaces X in the collection.*

Gromov's Compactness Theorem states that sequences of equibounded and equicompact metric spaces have a Gromov-Hausdorff converging subsequence [10]. In fact, Gromov proves a stronger version of this statement in [8]p 65: CHECK PAGE

Theorem 2.5 (Gromov's Compactness Theorem). *If a sequence of compact metric spaces, X_j , is equibounded and equicompact, then a subsequence of the X_j converges to a compact metric space X_∞ .*

Gromov also proved the following useful theorem:

Theorem 2.6. *If a sequence of compact metric spaces X_j converges to a compact metric space X_∞ then X_j are equibounded and equicompact. Furthermore, there is a compact metric spaces, Z , and isometric embeddings $\varphi_j : X_j \rightarrow Z$ such that*

$$(4) \quad d_H^Z(\varphi_j(X_{j_i}), \varphi_\infty(X_\infty)) \leq 2d_{GH}(X_j, X_\infty) \rightarrow 0.$$

This theorem allows one to define converging sequences of points:

Definition 2.7. We say that $x_j \in X_j$ converges to $x_\infty \in X_\infty$, if there is a common space Z as in Theorem 2.6 such that $\varphi_j(x_j) \rightarrow \varphi_\infty(x)$ as points in Z . If one discusses the limits of multiple sequences of points then one uses a common Z to determine the convergence to avoid difficulties arising from isometries in the limit space. Then one immediately has

$$(5) \quad \lim_{j \rightarrow \infty} d_{X_j}(x_j, x'_j) = d_{X_\infty}(x_\infty, x'_\infty)$$

whenever $x_j \rightarrow x_\infty$ and $x'_j \rightarrow x'_\infty$ via a common Z .

Theorem 2.6 also allows one to extend the Arzela-Ascoli Theorem:

Definition 2.8. A collection of functions, $f_j : X_j \rightarrow X'_j$ is said to be equicontinuous if for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ independent of j such that

$$(6) \quad f_j(B_x(\delta_\epsilon)) \subset B_{f_j(x)}(\epsilon) \quad \forall x \in X_j.$$

Theorem 2.9. Gromov-Arz-Asc Suppose X_j and X'_j are compact metric spaces converging in the Gromov-Hausdorff sense to compact metric spaces X_∞ and X'_∞ , and suppose $f_j : X_j \rightarrow X'_j$ are equicontinuous, then a subsequence of the f_j converge to a continuous function $f_\infty : X_\infty \rightarrow X'_\infty$ such that for any sequence $x_j \rightarrow x_\infty$ via a common Z we have $f_j(x_j) \rightarrow f_\infty(x_\infty)$.

In particular, one can define limits of curves $C_i : [0, 1] \rightarrow X_i$ (parametrized proportional to arclength with a uniform upper bound on length) to obtain curves $C_\infty : [0, 1] \rightarrow X_\infty$. So that when X_i are compact length spaces whose distances are achieved by minimizing geodesics, so are the limit spaces X_∞ .

One only needs uniform lower bounds on Ricci curvature and upper bounds on diameter to prove equicompactness on a sequence of Riemannian manifolds. This is a consequence of the Bishop-Gromov Volume Comparison Theorem [10]. Colding and Cheeger-Colding have studied the limits of such sequences of spaces proving volume convergence and eigenvalue convergence and many other interesting properties [5] [3]-[4]. One property of particular interest here, is that when the sequence of manifolds is noncollapsing (i.e. is assumed to have a uniform lower bound on volume), Cheeger-Colding prove that the limit space is countably \mathcal{H}^m rectifiable with the same dimension as the sequence [3].

Greene-Petersen have shown that conditions on contractibility and uniform upper bounds on diameter also suffice to achieve compactness without any assumption on Ricci curvature or volume [7]. Sormani-Wenger have shown that if one assumes a uniform linear contractibility function on the sequence of manifolds then the limit spaces achieved in their setting are also countably \mathcal{H}^m rectifiable with the same dimension as the sequence. Without the assumption of linearity, Schul-Wenger have provided an example where the Gromov-Hausdorff limit is not countably \mathcal{H}^m rectifiable. [19]. The proofs here involve the Intrinsic Flat Convergence.

2.2. Review of Ambrosio-Kirchheim Currents on Metric Spaces. In [1], Ambrosio-Kirchheim extend Federer-Fleming's notion of integral currents using DiGeorgi's notion of tuples of functions. Here we review their ideas. Here Z denotes a complete metric space.

In Federer-Fleming currents were defined as linear functionals on differential forms [6]. This approach extends naturally to smooth manifolds but not to complete metric spaces which do not have differential forms. In the place of differential forms, Ambrosio-Kirchheim use DiGeorgi's $m + 1$ tuples, $\omega \in \mathcal{D}^m(Z)$,

$$(7) \quad \omega = f\pi = (f, \pi_1 \dots \pi_m) \in \mathcal{D}^m(Z)$$

where $f : X \rightarrow \mathbb{R}$ is a bounded Lipschitz function and $\pi_i : X \rightarrow \mathbb{R}$ are Lipschitz.

In [1] Definitions 2.1, 2.2, 2.6 and 3.1, an m dimensional current $T \in \mathbf{M}_m(Z)$ is defined as follows:

Definition 2.10. *On a complete metric space, Z , an m dimensional **current**, denoted $T \in \mathbf{M}_m(Z)$, is a real valued multilinear functional on $\mathcal{D}^m(Z)$, with the following required properties:*

i) Locality:

$$(8) \quad T(f, \pi_1, \dots, \pi_m) = 0 \text{ if } \exists i \in \{1, \dots, m\} \text{ s.t. } \pi_i \text{ is constant on a nbd of } \{f \neq 0\}.$$

ii) Continuity of T with respect to the pointwise convergence of the π_i such that $\text{Lip}(\pi_i) \leq 1$.

iii) Finite mass: there exists a finite Borel measure μ on Z such that

$$(9) \quad |T(f, \pi_1, \dots, \pi_m)| \leq \prod_{i=1}^m \text{Lip}(\pi_i) \int_Z |f| d\mu \quad \forall (f, \pi_1, \dots, \pi_m) \in \mathcal{D}^m(Z).$$

Definition 2.11. [1][Defn 2.6] *The mass measure $\|T\|$ of a current $T \in \mathbf{M}_m(Z)$, is the smallest Borel measure μ such that*

$$(10) \quad |T(f, \pi)| \leq \int_X |f| d\mu \quad \forall (f, \pi) \text{ where } \text{Lip}(\pi_i) \leq 1.$$

The mass of T is defined

$$(11) \quad M(T) = \|T\|(Z) = \int_Z d\|T\|.$$

In particular

$$(12) \quad |T(f, \pi_1, \dots, \pi_m)| \leq \mathbf{M}(T) |f|_\infty \text{Lip}(\pi_1) \cdots \text{Lip}(\pi_m).$$

Stronger versions of locality and continuity, as well as product and chain rules are proven in [1][Theorem 3.5]. In particular they define $T(f, \pi_1, \dots, \pi_m)$ for f which are only Borel functions as limits of $T(f_j, \pi_1, \dots, \pi_m)$ where f_j are bounded Lipschitz functions converging to f in $L^1(E, \|T\|)$. They also prove

$$(13) \quad T(f, \pi_{\sigma(1)}, \dots, \pi_{\sigma(m)}) = \text{sgn}(\sigma) T(f, \pi_1, \dots, \pi_m)$$

for any permutation, σ , of $\{1, 2, \dots, m\}$.

Definition 2.12. [1][Defn 2.5] *The restriction $T \llcorner \omega \in \mathbf{M}_m(Z)$ of a current $T \in \mathbf{M}_{m+k}(Z)$ by a $k+1$ tuple $\omega = (g, \tau_1, \dots, \tau_k) \in \mathcal{D}^k(Z)$:*

$$(14) \quad (T \llcorner \omega)(f, \pi_1, \dots, \pi_m) := T(f \cdot g, \tau_1, \dots, \tau_k, \pi_1, \dots, \pi_m).$$

Given a Borel set, A ,

$$(15) \quad T \llcorner A := T \llcorner \omega$$

where $\omega = 1_A \in \mathcal{D}^0(Z)$ is the indicator function of the set. In this case,

$$(16) \quad \mathbf{M}(T \llcorner \omega) = \|T\|(A).$$

Ambrosio-Kirchheim then define:

Definition 2.13. *Given a Lipschitz map $\varphi : Z \rightarrow Z'$, the push forward of a current $T \in \mathbf{M}_m(Z)$ to a current $\varphi_\# T \in \mathbf{M}_m(Z')$ is given in [1][Defn 2.4] by*

$$(17) \quad \varphi_\# T(f, \pi_1, \dots, \pi_m) := T(f \circ \varphi, \pi_1 \circ \varphi, \dots, \pi_m \circ \varphi)$$

exactly as in the smooth setting.

Remark 2.14. *One immediately sees that*

$$(18) \quad (\varphi_{\#}T) \llcorner (f, \pi_1, \dots, \pi_k) = \varphi_{\#}(T \llcorner (f \circ \varphi, \pi_1 \circ \varphi, \dots, \pi_k \circ \varphi))$$

and

$$(19) \quad (\varphi_{\#}T) \llcorner A = (\varphi_{\#}T) \llcorner (1_A) = \varphi_{\#}(T \llcorner (1_A \circ \varphi)) = \varphi_{\#}(T \llcorner \varphi^{-1}(A)).$$

In (2.4) [1], Ambrosio-Kirchheim show that

$$(20) \quad \|\varphi_{\#}T\| \leq [\text{Lip}(\varphi)]^m \varphi_{\#}\|T\|,$$

so that when φ is an isometric embedding

$$(21) \quad \|\varphi_{\#}T\| = \varphi_{\#}\|T\| \text{ and } \mathbf{M}(T) = \mathbf{M}(\varphi_{\#}T).$$

The simplest example of a current is:

Example 2.15. *If one has a bi-Lipschitz map, $\varphi : \mathbb{R}^m \rightarrow Z$, and a Lebesgue function $h \in L^1(A, \mathbb{Z})$ where $A \in \mathbb{R}^m$ is Borel, then $\varphi_{\#}[h] \in \mathbf{M}_m(Z)$ an m dimensional current in Z . Note that*

$$(22) \quad \varphi_{\#}[h](f, \pi_1, \dots, \pi_m) = \int_{A \subset \mathbb{R}^m} (h \circ \varphi)(f \circ \varphi) d(\pi_1 \circ \varphi) \wedge \dots \wedge d(\pi_m \circ \varphi)$$

where $d(\pi_i \circ \varphi)$ is well defined almost everywhere by Rademacher's Theorem. Here the mass measure is

$$(23) \quad \| [h] \| = h d\mathcal{L}_m$$

and the mass is

$$(24) \quad \mathbf{M}([h]) = \int_A h d\mathcal{L}_m.$$

In [1][Theorem 4.6] Ambrosio-Kirchheim define a canonical set associated with any integer rectifiable current:

Definition 2.16. *The (canonical) set of a current, T , is the collection of points in Z with positive lower density:*

$$(25) \quad \text{set}(T) = \{p \in Z : \Theta_{*m}(\|T\|, p) > 0\},$$

where the definition of lower density is

$$(26) \quad \Theta_{*m}(\mu, p) = \liminf_{r \rightarrow 0} \frac{\mu(B_p(r))}{\omega_m r^m}.$$

In [1] Definition 4.2 and Theorems 4.5-4.6, an integer rectifiable current is defined using the Hausdorff measure, \mathcal{H}^m :

Definition 2.17. *Let $m \geq 1$. A current, $T \in \mathcal{D}_m(Z)$, is rectifiable if $\text{set}(T)$ is countably \mathcal{H}^m rectifiable and if $\|T\|(A) = 0$ for any set A such that $\mathcal{H}^k(A) = 0$. We write $T \in \mathcal{R}_m(Z)$.*

We say $T \in \mathcal{R}_m(Z)$ is integer rectifiable, denoted $T \in \mathcal{I}_m(Z)$, if for any $\varphi \in \text{Lip}(Z, \mathbb{R}^m)$ and any open set $A \in Z$, we have $\varphi_{\#}(T \llcorner A) = [\theta]$ for some $\theta \in \mathcal{L}^1(\mathbb{R}^k, Z)$. In fact, $T \in \mathcal{I}_m(Z)$ iff it has a parametrization. A parametrization $(\{\varphi_i\}, \{\theta_i\})$ of an integer rectifiable current $T \in \mathcal{I}^m(Z)$ is a collection of bi-Lipschitz maps $\varphi_i : A_i \rightarrow Z$ with $A_i \subset \mathbb{R}^m$ precompact Borel measurable and with pairwise disjoint images and weight functions $\theta_i \in L^1(A_i, \mathbb{N})$ such that

$$(27) \quad T = \sum_{i=1}^{\infty} \varphi_{i\#}[\theta_i] \quad \text{and} \quad \mathbf{M}(T) = \sum_{i=1}^{\infty} \mathbf{M}(\varphi_{i\#}[\theta_i]).$$

A 0 dimensional rectifiable current is defined by the existence of countably many distinct points, $\{x_i\} \in Z$, weights $\theta_i \in \mathbb{R}^+$ and orientation, $\sigma_i \in \{-1, +1\}$ such that

$$(28) \quad T(f) = \sum_h \sigma_i \theta_i f(x_i) \quad \forall f \in \mathcal{B}^\infty(Z).$$

where $\mathcal{B}^\infty(Z)$ is the class of bounded Borel functions on Z and where

$$(29) \quad \mathbf{M}(T) = \sum_h \theta_i < \infty$$

If T is integer rectifiable $\theta_i \in \mathbb{Z}^+$, so the sum must be finite.

In particular, the mass measure of $T \in \mathbf{I}_m(Z)$ satisfies

$$(30) \quad \|T\| = \sum_{i=1}^{\infty} \|\varphi_{i\#}[\theta_i]\|.$$

Theorems 4.3 and 8.8 of [1] provide necessary and sufficient criteria for determining when a current is integer rectifiable.

Note that the current in Example 2.15 is an integer rectifiable current.

Example 2.18. If one has a Riemannian manifold, M^m , and a biLipschitz map $\varphi : M^m \rightarrow Z$, then $T = \varphi_{\#}[1_M]$ is an integer rectifiable current of dimension m in Z . If φ is an isometric embedding, and $Z = M$ then $\mathbf{M}(T) = \text{Vol}(M^m)$. Note further that $\text{set}(T) = \varphi(M)$.

If M has a conical singularity then $\text{set}(T) = \varphi(M)$. However, if M has a cusp singularity at a point $p \in M$ then $\text{set}(T) = \varphi(M \setminus \{p\})$.

Definition 2.19. [1][Defn 2.3] The boundary of $T \in \mathbf{M}_m(Z)$ is defined

$$(31) \quad \partial T(f, \pi_1, \dots, \pi_{m-1}) := T(1, f, \pi_1, \dots, \pi_{m-1}) \in M_{m-1}(Z)$$

When $m = 0$, we set $\partial T = 0$.

Note that $\varphi_{\#}(\partial T) = \partial(\varphi_{\#}T)$.

Definition 2.20. [1][Defn 3.4 and 4.2] An integer rectifiable current $T \in \mathbf{I}_m(Z)$ is called an integral current, denoted $T \in \mathbf{I}_m(Z)$, if ∂T defined as

$$(32) \quad \partial T(f, \pi_1, \dots, \pi_{m-1}) := T(1, f, \pi_1, \dots, \pi_{m-1})$$

has finite mass. The total mass of an integral current is

$$(33) \quad \mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T).$$

Observe that $\partial \partial T = 0$. In [1] Theorem 8.6, Ambrosio-Kirchheim prove that

$$(34) \quad \partial : \mathbf{I}_m(Z) \rightarrow \mathbf{I}_{m-1}(Z)$$

whenever $m \geq 1$. By (20) one can see that if $\varphi : Z_1 \rightarrow Z_2$ is Lipschitz, then

$$(35) \quad \varphi_{\#} : \mathbf{I}_m(Z_1) \rightarrow \mathbf{I}_m(Z_2).$$

Remark 2.21. If T is an \mathcal{H}^1 integral current then ∂T is an \mathcal{H}^0 integer rectifiable current so $H = \text{set} \partial T$ must be finite and $\theta_{p_h} = \|\partial T\|(p_h) \in \mathbb{Z}^+$ for all $p \in H$ and

$$(36) \quad \partial T(f) = \sum_{h \in H} \sigma_h \theta_h f(p_h) \quad \forall f \in \mathcal{B}^\infty(Z).$$

as described above. In addition, we have

$$(37) \quad 0 = T(1, 1) = \partial T(1) = \sum_{h \in H} \sigma_h \theta_h.$$

Example 2.22. If T is an \mathcal{H}^1 rectifiable current then

$$(38) \quad T = \sum_{i=1}^{\infty} \sigma_i \theta_i \varphi_{i\#} [\chi_{A_i}]$$

where $\theta_i \in \mathbb{Z}^+$, $\sigma_i \in \{+1, -1\}$ and A_i is an interval with $\bar{A}_i = [a_i, b_i]$ because all Borel sets are unions of intervals and all integer valued Borel functions can be written up to Lebesgue measure 0 as a countable sum of characteristic functions of intervals. One might like to write:

$$(39) \quad \partial T(f) = \sum_i \sigma_i \theta_i (f(\varphi_i(b_i)) - f(\varphi_i(a_i))) \quad \forall f \in \mathcal{B}^\infty(Z).$$

This works when the sum happens to be a finite sum. Yet if T is a infinite collection of circles based at a common point, $(0, 0) \in \mathbb{R}^2$, defined with $\sigma_i = 1$, $\theta_i = 1$, $A_i = [0, \pi]$ and

$$(40) \quad \varphi_i(s) = (r_i \cos(s) + r_i, r_i \sin(s)) \text{ for } i \text{ odd and}$$

$$(41) \quad \varphi_i(s) = (r_i \cos(s + \pi) + r_i, r_i \sin(s + \pi)) \text{ for } i \text{ even}$$

where $r_{2i} = r_{2i-1} = 1/i$ then

$$(42) \quad \varphi_i(a_i) = (2r_i, 0) \quad \text{and} \quad \varphi_i(b_i) = (0, 0) \text{ for } i \text{ odd and}$$

$$(43) \quad \varphi_i(a_i) = (0, 0) \quad \text{and} \quad \varphi_i(b_i) = (2r_i, 0) \text{ for } i \text{ even.}$$

So when $f(0, 0) = 1$, we end up with an infinite sum whose terms are all $+1$ and -1 .

2.3. Review of Ambrosio-Kirchheim Slicing Theorems. As in Federer-Fleming, Ambrosio-Kirchheim consider the slices of currents:

Theorem 2.23. [Ambrosio-Kirchheim] [1]/Theorems 5.6-5.7] Let Z be a complete metric space, $T \in \mathbf{I}_m Z$ and $f : Z \rightarrow \mathbb{R}$ a Lipschitz function. Let

$$(44) \quad \langle T, f, s \rangle := \partial (T \llcorner f^{-1}(-\infty, s]) - (\partial T) \llcorner f^{-1}(-\infty, s]),$$

so that $\text{set}(\langle T, f, s \rangle) \subset (\text{set}(T) \cup \text{set}(\partial T)) \cap f^{-1}(s)$,

$$(45) \quad \partial \langle T, f, s \rangle = -\partial T, f, s \rangle$$

and $\langle T_1 + T_2, f, s \rangle = \langle T_1, f, s \rangle + \langle T_2, f, s \rangle$. Then for almost every slice $s \in \mathbb{R}$, $\langle T, f, s \rangle$ is an integral current and we can integrate the masses to obtain:

$$(46) \quad \int_{s \in \mathbb{R}} \mathbf{M}(\langle T, f, s \rangle) ds = \mathbf{M}(T \llcorner df) \leq \text{Lip}(f) \mathbf{M}(T)$$

where

$$(47) \quad (T \llcorner df)(h, \pi_1, \dots, \pi_{m-1}) = T(h, f, \pi_1, \dots, \pi_{m-1}).$$

In particular, for almost every $s > 0$ one has

$$(48) \quad T \llcorner f^{-1}(-\infty, s] \in \mathbf{I}_{m-1}(Z).$$

Furthermore for all Borel sets A we have

$$(49) \quad \langle T \llcorner A, f, s \rangle = \langle T, f, s \rangle \llcorner A$$

and

$$(50) \quad \int_{s \in \mathbb{R}} \|\langle T, f, s \rangle\|(A) ds = \|T \llcorner df\|(A).$$

Remark 2.24. Observe that for any $T \in \mathbf{I}_m(Z')$, and any Lipschitz functions, $\varphi : Z \rightarrow Z'$ and $f : Z' \rightarrow \mathbb{R}$ and any $s > 0$, we have

$$\begin{aligned}
 (51) \quad \langle \varphi_{\#} T, f, s \rangle &= \partial \left((\varphi_{\#} T) \llcorner f^{-1}(-\infty, s] \right) - (\partial \varphi_{\#} T) \llcorner f^{-1}(-\infty, s] \\
 (52) &= \partial \left(\varphi_{\#} (T \llcorner \varphi^{-1}(f^{-1}(-\infty, s])) \right) - (\varphi_{\#} \partial T) \llcorner f^{-1}(-\infty, s] \\
 (53) &= \partial \left(\varphi_{\#} (T \llcorner (f \circ \varphi)^{-1}(-\infty, s]) \right) - \varphi_{\#} \left(\partial T \llcorner \varphi^{-1}(f^{-1}(-\infty, s]) \right) \\
 (54) &= \left(\varphi_{\#} \partial (T \llcorner (f \circ \varphi)^{-1}(-\infty, s]) \right) - \varphi_{\#} \left(\partial T \llcorner (f \circ \varphi)^{-1}(-\infty, s]) \right) \\
 (55) &= \varphi_{\#} \langle T, (f \circ \varphi), s \rangle
 \end{aligned}$$

Remark 2.25. Ambrosio-Kirchheim then iterate this definition, $f_i : Z \rightarrow \mathbb{R}$, $s_i \in \mathbb{R}$, to define iterated slices:

$$(56) \quad \langle T, f_1, \dots, f_k, s_1, \dots, s_k \rangle = \langle \langle T, f_1, \dots, f_{k-1}, s_1, \dots, s_{k-1} \rangle, f_k, s_k \rangle,$$

so that

$$(57) \quad \langle T_1 + T_2, f_1, \dots, f_k, s_1, \dots, s_k \rangle = \langle T_1, f_1, \dots, f_k, s_1, \dots, s_k \rangle + \langle T_2, f_1, \dots, f_k, s_1, \dots, s_k \rangle.$$

In [1] Lemma 5.9 they prove,

$$(58) \quad \langle T, f_1, \dots, f_k, s_1, \dots, s_k \rangle = \langle \langle T, f_1, \dots, f_i, s_1, \dots, s_i \rangle, f_{i+1}, \dots, f_k, s_{i+1}, \dots, s_k \rangle.$$

In [1] (5.9) they prove,

$$(59) \quad \int_{\mathbb{R}^k} \|\langle T, f_1, \dots, f_k, s_1, \dots, s_k \rangle\| ds_1 \dots ds_k = \|T \llcorner (1, f_1, \dots, f_k)\|,$$

where

$$(60) \quad (T \llcorner df)(h, \pi_1, \dots, \pi_{m-k}) = T(h, f_1, \dots, f_k, \pi_1, \dots, \pi_{m-k}),$$

so

$$(61) \quad \int_{\mathbb{R}^k} \mathbf{M}(\langle T, f_1, \dots, f_k, s_1, \dots, s_k \rangle) \mathcal{L}^k = \mathbf{M}(T \llcorner df) \leq \prod_{j=1}^k \text{Lip}(f_j) \mathbf{M}(T).$$

In [1] (5.15) they prove for any Borel set $A \subset Z$ and \mathcal{L}^m almost every $(s_1, \dots, s_k) \in \mathbb{R}^k$,

$$(62) \quad \langle T \llcorner A, f_1, \dots, f_k, s_1, \dots, s_k \rangle = \langle t, f_1, \dots, f_k, s_1, \dots, s_k \rangle \llcorner A.$$

and

$$(63) \quad \int_{s \in \mathbb{R}^k} \|\langle T, f_1, \dots, f_k, s_1, \dots, s_k \rangle\|(A) ds = \|T \llcorner df\|(A).$$

By (64) one can easily prove by induction that

$$(64) \quad \partial \langle T, f_1, \dots, f_k, s_1, \dots, s_k \rangle = (-1)^k \langle \partial T, f_1, \dots, f_k, s_1, \dots, s_k \rangle.$$

In [1] Theorem 5.7 they prove

$$(65) \quad \langle T, f_1, \dots, f_k, s_1, \dots, s_k \rangle \in \mathbf{I}_{m-k}(Z).$$

for \mathcal{L}^k almost every $(s_1, \dots, s_k) \in \mathbb{R}^k$. By Remark 2.24 one can prove inductively that

$$(66) \quad \langle \varphi_{\#} T, f_1, \dots, f_k, s_1, \dots, s_k \rangle = \varphi_{\#} \langle T, f_1 \circ \varphi, \dots, f_k \circ \varphi, s_1, \dots, s_k \rangle.$$

2.4. Review of Convergence of Currents. Ambrosio Kirchheim's Compactness Theorem, which extends Federer-Fleming's Flat Norm Compactness Theorem, is stated in terms of weak convergence of currents. See Definition 3.6 in [1] which extends Federer-Fleming's notion of weak convergence except that they do not require compact support.

Definition 2.26. A sequence of integral currents $T_j \in \mathbf{I}_m(Z)$ is said to converge weakly to a current T iff the pointwise limits satisfy

$$(67) \quad \lim_{j \rightarrow \infty} T_j(f, \pi_1, \dots, \pi_m) = T(f, \pi_1, \dots, \pi_m)$$

for all bounded Lipschitz $f : Z \rightarrow \mathbb{R}$ and Lipschitz $\pi_i : Z \rightarrow \mathbb{R}$. We write

$$(68) \quad T_j \rightarrow T$$

One sees immediately that $T_j \rightarrow T$ implies

$$(69) \quad \partial T_j \rightarrow \partial T,$$

$$(70) \quad \varphi_{\#} T_j \rightarrow \varphi_{\#} T$$

and

$$(71) \quad T_j \llcorner (f, \pi_1, \dots, \pi_k) \rightarrow T \llcorner (f, \pi_1, \dots, \pi_k).$$

However $T_j \llcorner A$ need not converge weakly to $T \llcorner A$ as seen in the following example:

Example 2.27. Let $Z = \mathbb{R}^2$ with the Euclidean metric. Let $\varphi_j : [0, 1] \rightarrow Z$ be $\varphi_j(t) = (1/j, t)$ and $\varphi_{\infty}(t) = (0, t)$. Let $S \in \mathbf{I}_1([0, 1])$ be

$$(72) \quad S(f, \pi_1) = \int_0^1 f d\pi_1$$

Let $T_j \in \mathbf{I}_1(Z)$ be defined $T_j = \varphi_{j\#}(S)$. Then $T_j \rightarrow T_{\infty}$. Taking $A = [0, 1] \times (0, 1)$, we see that $T_j \llcorner A = T_j$ but $T_{\infty} \llcorner A = 0$.

Immediately below the definition of weak convergence [1] Defn 3.6, Ambrosio-Kirchheim prove the lower semicontinuity of mass:

Remark 2.28. If T_j converges weakly to T , then $\liminf_{j \rightarrow \infty} \mathbf{M}(T_j) \geq \mathbf{M}(T)$.

Theorem 2.29 (Ambrosio-Kirchheim Compactness). Given any complete metric space Z , a compact set $K \subset Z$ and $A_0, V_0 > 0$. Given any sequence of integral currents $T_j \in \mathbf{I}_m(Z)$ satisfying

$$(73) \quad \mathbf{M}(T_j) \leq V_0, \mathbf{M}(\partial T_j) \leq A_0 \text{ and } \text{set}(T_j) \subset K,$$

there exists a subsequence, T_{j_i} , and a limit current $T \in \mathbf{I}_m(Z)$ such that T_{j_i} converges weakly to T .

2.5. Review of Integral Current Spaces. The notion of an integral current space was introduced by the author and Stefan Wenger in [20]:

Definition 2.30. An m dimensional metric space (X, d, T) is called an integral current space if it has a integral current structure $T \in \mathbf{I}_m(\bar{X})$ where \bar{X} is the metric completion of X and $\text{set}(T) = X$. Given an integral current space $M = (X, d, T)$ we will use $\text{set}(M)$ or X_M to denote X , $d_M = d$ and $\llbracket M \rrbracket = T$.

Note that $\text{set}(\partial T) \subset \bar{X}$. The boundary of (X, d, T) is then the integral current space:

$$(74) \quad \partial(X, d_X, T) := (\text{set}(\partial T), d_{\bar{X}}, \partial T).$$

If $\partial T = 0$ then we say (X, d, T) is an integral current without boundary.

Remark 2.31. Note that any m dimensional integral current space is countably \mathcal{H}^m rectifiable with orientated charts, φ_i and weights θ_i provided as in (27). A 0 dimensional integral current space is a finite collection of points with orientations, σ_i and weights θ_i provided as in (28). If this space is the boundary of a 1 dimensional integral current space, then as in Remark 2.21, the sum of the signed weights is 0.

Example 2.32. A compact oriented Riemannian manifold with boundary, M^m , is an integral current space, where $X = M^m$, d is the standard metric on M and T is integration over M . In this case $\mathbf{M}(M) = \text{Vol}(M)$ and ∂M is the boundary manifold. When M has no boundary, $\partial M = 0$.

Definition 2.33. The space of $m \geq 0$ dimensional integral current spaces, \mathcal{M}^m , consists of all metric spaces which are integral current spaces with currents of dimension m as in Definition 2.30 as well as the $\mathbf{0}$ spaces. Then $\partial : \mathcal{M}^{m+1} \rightarrow \mathcal{M}^m$.

Remark 2.34. A 0 dimensional integral current space, $M = (X, d, T)$, is a finite collection of points, $\{p_1, \dots, p_N\}$, with a metric $d_{i,j} = d(p_i, p_j)$ and a current structure defined by assigning a weight, $\theta_i \in \mathbb{Z}^+$, and an orientation, $\sigma_i \in \{+1, -1\}$ to each $p_i \in X$ and

$$(75) \quad \mathbf{M}(M) = \sum_{i=1}^N \theta_i.$$

If M is the boundary of a 1 dimensional integral current space then, as in Remark 2.21, we have

$$(76) \quad \sum_{i=1}^N \sigma_i \theta_i = 0$$

In particular $N \geq 2$ if $M \neq \mathbf{0}$.

2.6. Review of the Intrinsic Flat distance. The Intrinsic Flat distance was defined in work of the author and Stefan Wenger [20] as a new distance between Riemannian manifolds based upon the work of Ambrosio-Kirchheim reviewed above.

Recall that the flat distance between m dimensional integral currents $S, T \in \mathbf{I}_m(Z)$ is given by

$$(77) \quad d_F^Z(S, T) := \inf\{\mathbf{M}(U) + \mathbf{M}(V) : S - T = U + \partial V\}$$

where $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$. This notion of a flat distance was first introduced by Whitney in [23] and later adapted to rectifiable currents by Federer-Fleming [6]. The flat distance between Ambrosio-Kirchheim's integral currents was studied by Wenger in [21]. In particular, Wenger proved that if $T_j \in \mathbf{I}_m(Z)$ has $\mathbf{M}(T_j) \leq V_0$ and $\mathbf{M}(\partial T_j) \leq A_0$ then

$$(78) \quad T_j \rightarrow T \text{ iff } d_F^Z(T_j, T) \rightarrow 0$$

exactly as in Federer-Fleming.

The intrinsic flat distance between integral current spaces was first defined in [20][Defn 1.1]:

Definition 2.35. For $M_1 = (X_1, d_1, T_1)$ and $M_2 = (X_2, d_2, T_2) \in \mathcal{M}^m$ let the intrinsic flat distance be defined:

$$(79) \quad d_{\mathcal{F}}(M_1, M_2) := \inf d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2),$$

where the infimum is taken over all complete metric spaces (Z, d) and isometric embeddings $\varphi_1 : (\bar{X}_1, d_1) \rightarrow (Z, d)$ and $\varphi_2 : (\bar{X}_2, d_2) \rightarrow (Z, d)$ and the flat norm d_F^Z is taken in Z . Here \bar{X}_i

denotes the metric completion of X_i and d_i is the extension of d_i on \bar{X}_i , while $\phi_{\#}T$ denotes the push forward of T .

In [20], it is observed that

$$(80) \quad d_{\mathcal{F}}(M_1, M_2) \leq d_{\mathcal{F}}(M_1, 0) + d_{\mathcal{F}}(0, M_2) \leq \mathbf{M}(M_1) + \mathbf{M}(M_2).$$

There it is also proven that $d_{\mathcal{F}}$ satisfies the triangle inequality [20][Thm 3.2] and is a distance:

Theorem 2.36. [20][Thm 3.27] *Let M, N be precompact integral current spaces and suppose that $d_{\mathcal{F}}(M, N) = 0$. Then there is a current preserving isometry from M to N where an isometry $f : X_M \rightarrow X_N$ is called a current preserving isometry between M and N , if its extension $\bar{f} : \bar{X}_M \rightarrow \bar{X}_N$ pushes forward the current structure on M to the current structure on N : $\bar{f}_{\#}T_M = T_N$*

In [20] Theorem 3.23 it is also proven that

Theorem 2.37. [20][Thm 4.23] *Given a pair of precompact integral current spaces, $M_1^m = (X_1, d_1, T_1)$ and $M_2^m = (X_2, d_2, T_2)$, there exists a compact metric space, (Z, d_Z) , integral currents $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$, and isometric embeddings $\varphi_1 : \bar{X}_1 \rightarrow Z$ and $\varphi_2 : \bar{X}_2 \rightarrow Z$ with*

$$(81) \quad \varphi_{\#}T_1 - \varphi'_{\#}T_2 = U + \partial V$$

such that

$$(82) \quad d_{\mathcal{F}}(M_1, M_2) = \mathbf{M}(U) + \mathbf{M}(V).$$

Remark 2.38. *The metric space Z in Theorem 2.37 has*

$$(83) \quad \text{Diam}(Z) \leq 3 \text{Diam}(X_1) + 3 \text{Diam}(X_2).$$

This is seen by consulting the proof of Theorem 3.23 in [20], where Z is constructed as the injective envelope of the Gromov-Hausdorff limit of a sequence of spaces Z_n with this same diameter bound.

The following theorem in [20] is an immediate consequence of Gromov and Ambrosio-Kirchheim's Compactness Theorems:

Theorem 2.39. *Given a sequence of m dimensional integral current spaces $M_j = (X_j, d_j, T_j)$ such that X_j are equibounded and equicompact and with uniform upper bounds on mass and boundary mass. A subsequence converges in the Gromov-Hausdorff sense $(X_{j_i}, d_{j_i}) \xrightarrow{GH} (Y, d_Y)$ and in the intrinsic flat sense $(X_{j_i}, d_{j_i}, T_{j_i}) \xrightarrow{\mathcal{F}} (X, d, T)$ where either (X, d, T) is an m dimensional integral current space with $X \subset Y$ or it is the $\mathbf{0}$ current space.*

Immediately one notes that if Y has Hausdorff dimension less than m , then $(X, d, T) = \mathbf{0}$. In [20] Example A.7, there is an example where M_j are compact three dimensional Riemannian manifolds with positive scalar curvature that converge in the Gromov-Hausdorff sense to a standard three sphere but in the Intrinsic Flat sense to $\mathbf{0}$. It is proven in [19], that if (X_j, d_j, T_j) are compact Riemannian manifolds with nonnegative Ricci curvature or a uniform linear contractibility function, then the intrinsic flat and Gromov-Hausdorff limits agree.

There are many examples of sequences of Riemannian manifolds which have no Gromov-Hausdorff limit but have an intrinsic flat limit. The first is Ilmanen's Example of an increasingly hairy three sphere with positive scalar curvature described in [20] Example A.7.

Other examples appear in work of the author with Dan Lee concerning the stability of the Positive Mass Theorem [13] [14] and in work of the author with Sajjad Lakzian concerning smooth convergence away from singular sets [12].

The following three theorems are proven in work of the author with Wenger [20]. Combining these theorems with the work of Ambrosio-Kirchheim reviewed earlier will lead to many of the properties of Intrinsic Flat Convergence described in this paper:

Theorem 2.40. [20][Thm 4.2] *If a sequence of integral current spaces $M_j = (X_j, d_j, T_j)$ converges in the intrinsic flat sense to an integral current space, $M_0 = (X_0, d_0, T_0)$, then there is a separable complete metric space, Z , and isometric embeddings $\varphi_j : X_j \rightarrow Z$ such that $\varphi_{\#}T_j$ flat converges to $\varphi_{0\#}T_0$ in Z and thus converge weakly as well.*

Theorem 2.41. [20][Thm 4.3] *If a sequence of integral current spaces $M_j = (X_j, d_j, T_j)$ converges in the intrinsic flat sense to the zero integral current space, $\mathbf{0}$, then we may choose points $x_j \in X_j$ and a separable complete metric space, Z , and isometric embeddings $\varphi_j : X_j \rightarrow Z$ such that $\varphi_j(x_j) = z_0 \in Z$ and $\varphi_{\#}T_j$ flat converges to 0 in Z and thus converges weakly as well.*

Theorem 2.42. *If a sequence of integral current spaces M_j converges in the intrinsic flat sense to a integral current space, M_∞ , then*

$$(84) \quad \liminf_{i \rightarrow \infty} \mathbf{M}(M_i) \geq \mathbf{M}(M_\infty)$$

Proof. This follows from Theorems 2.40 and 2.41 combined with Ambrosio-Kirchheim's lower semicontinuity of mass [c.f. Remark 2.42]. \square

Finally there is Wenger's Compactness Theorem [22]:

Theorem 2.43 (Wenger). *Given $A_0, V_0, D_0 > 0$. If $M_j = (X_j, d_j, T_j)$ are integral current spaces such that*

$$(85) \quad \text{Diam}(M_j) \leq D_0 \quad \mathbf{M}(M_j) \leq V_0 \quad \mathbf{M}(\partial(M_j)) \leq A_0$$

then a subsequence converges in the Intrinsic Flat Sense to an integral current space of the same dimension, possibly the $\mathbf{0}$ space.

Recall that this theorem applies to oriented Riemannian manifolds of the same dimension with a uniform upper bound on volume and a uniform upper bound on the volumes of the boundaries. One immediately sees that the conditions required to apply Wenger's Compactness Theorem are far weaker than the conditions required for Gromov's Compactness Theorem. The only difficulty lies in determining whether the limit space is $\mathbf{0}$ or not. Wenger's proof involves a thick thin decomposition, a study of filling volumes and uses the notion of an ultralimit.

It should be noted that Theorems 2.40 and ?? and all other theorems reviewed and proven within this paper are proven without applying Wenger's Compactness Theorem. Thus one may wish to attempt alternate proofs of Wenger's Compactness Theorem using the results in this paper.

2.7. Review of Filling Volumes. The notion of a filling volume was first introduced by Gromov in [9]. Wenger studied the filling volumes of integral currents in metric spaces in [21]

First we discuss the Plateau Problem on complete metric spaces. Given a integral current $T \in \mathbf{I}_m Z$, one may define the filling volume of ∂T within Z as

$$(86) \quad \text{FillVol}_Z(T) = \inf\{\mathbf{M}(S) : S \in \mathbf{I}_m(Z) \text{ s.t. } \partial S = \partial T\}.$$

This immediately provides an upper bound on the flat distance:

$$(87) \quad d_{\mathcal{F}}^Z(\partial T, \mathbf{0}) \leq \text{FillVol}_Z(\partial \mathbf{T}) \leq \mathbf{M}(\mathbf{T})$$

Ambrosio-Kirchheim proved this infimum is achieved on Banach spaces, Z [1] [Theorem 10.2].

Wenger defined the absolute filling volume of $T \in \mathbf{I}_m Y$ to be

$$(88) \quad \text{FillVol}_\infty(\partial T) = \inf\{\mathbf{M}(S) : S \in \mathbf{I}_m(Z) \text{ s.t. } \partial S = \varphi_\# \partial T\}$$

where the infimum is taken over all isometric embeddings $\varphi : Y \rightarrow Z$, all complete metric spaces, Z , and all $S \in \mathbf{I}_m(Z)$ such that $\partial S = \varphi_\# T$. Clearly

$$(89) \quad \text{FillVol}_\infty(\partial T) \leq \text{FillVol}_Y(\partial T).$$

He proved that this infimum is achieved and via the Kuratowski Embedding Theorem, this is achieved on a Banach space, Z [21].

Here we will use the following notion of a filling of an integral current space:

Definition 2.44. *Given an integral current space $M = (X, d, T) \in \mathcal{M}^m$ with $m \geq 1$ we define*

$$(90) \quad \text{FillVol}(\partial M) := \inf\{\mathbf{M}(N) : N \in \mathcal{M}^{m+1} \text{ and } \partial N = \partial M\}.$$

That is we require that there exists a current preserving isometry from ∂N onto ∂M , where as usual, we have taken the metrics on the boundary spaces to be the restrictions of the metrics on the metric completions of N and M respectively.

It is easy to see that

$$(91) \quad \text{FillVol}(\partial M) \leq \mathbf{M}(M).$$

and

$$(92) \quad d_{\mathcal{F}}(\partial M, \mathbf{0}) \leq \text{FillVol}(\partial M) \leq \mathbf{M}(M)$$

for any integral current space M .

Remark 2.45. *The infimum in the definition of the filling volume is achieved. This may be seen by imitating the proof that the infimum in the definition of the intrinsic flat norm is attained in [20]. Since the N achieving the infimum has $\partial N \neq \mathbf{0}$, the filling volume is positive.*

Any integral current space, $M = (X, d, T)$, is separable and so one can map the space into a Banach space, Z , via the Kuratowski Embedding theorem, $\iota : X \rightarrow Z$. By Ambrosio-Kirchheim's solution to the Plateau problem on Banach spaces [1][Prop 10.2],

$$(93) \quad \text{FillVol}(\partial M) \leq \text{FillVol}_Z(\varphi_\#(\partial T)) \leq \text{Diam}(X)\mathbf{M}(\partial T) \text{Diam}(M)\mathbf{M}(\partial M).$$

Wenger proved that the filling volume is continuous with respect to weak convergence (and thus also intrinsic flat convergence when applying Theorem 2.40). Here we provide a precise estimate which will be needed later in the paper:

Theorem 2.46. *For any pair of integral current spaces, M_i , we have*

$$(94) \quad \text{FillVol}(\partial M_1) \leq \text{FillVol}(\partial M_2) + d_{\mathcal{F}}(M_1, M_2).$$

and

$$(95) \quad \text{FillVol}(\partial M_1) \leq \text{FillVol}(\partial M_2) + (1 + 3 \text{Diam}(M_1) + 3 \text{Diam}(M_2))d_{\mathcal{F}}(\partial M_1, \partial M_2).$$

Proof. This can be seen because Theorem 2.37 and Remark 2.38 imply that there exists integral current spaces A, B such that and isometric embeddings, $\varphi_i : M_i \rightarrow Z$, such that

$$(96) \quad \varphi_{1\#}M_1 - \varphi_{2\#}M_2 = \partial B + A$$

where

$$(97) \quad d_{\mathcal{F}}(M_1, M_2) = \mathbf{M}(A) + \mathbf{M}(B)$$

In particular

$$(98) \quad \varphi_{1\#}\partial M_1 - \varphi_{2\#}\partial M_2 = \partial A$$

Next we need to fill in ∂M_2 and glue that integral current space to A . Add details.

Next observe that there exists integral current spaces M_3, M_4 and isometric embeddings, $\varphi_i : M_i \rightarrow Z'$, such that

$$(99) \quad \varphi_{1\#}\partial M_1 - \varphi_{2\#}\partial M_2 = \varphi_{3\#}\partial M_3 + \varphi_{4\#}M_4$$

where

$$(100) \quad d_{\mathcal{F}}(\partial M_1, \partial M_2) = \mathbf{M}(M_3) + \mathbf{M}(M_4).$$

By Remark 2.38,

$$(101) \quad \text{Diam}(M_3), \text{Diam}(M_4) \leq 3 \text{Diam}(M_1) + 3 \text{Diam}(M_2).$$

In [1] Theorem 10.2, it is proven that

$$(102) \quad \text{FillVol}(M_4) \leq \text{Diam}(M_4)\mathbf{M}(M_4).$$

So if we glue the integral current spaces, N_i , which achieve the filling volume of M_i for $i = 2, 4$, to M_3 as needed, then we see that

$$(103) \quad \text{FillVol}(\partial M_1) \leq \text{FillVol}(\partial M_2) + \mathbf{M}(M_3) + \text{FillVol}(M_4)$$

$$(104) \quad \leq \text{FillVol}(\partial M_2) + \mathbf{M}(M_3) + \text{Diam}(M_4)\mathbf{M}(M_4)$$

$$(105) \quad \leq \text{FillVol}(\partial M_2) + (\text{Diam}(M_4) + 1)(\mathbf{M}(M_3) + \mathbf{M}(M_4))$$

and we have our second claim. \square

Remark 2.47. Gromov's Filling Volume in [9] is defined as in (90) where N^{n+1} must be a Riemannian Manifold. Thus it is conceivable that the filling volume in Definition 2.44 might have a smaller value both because integral current spaces have integer weight and because we have a wider class of metrics to choose from, including metrics which are not length metrics.

3. METRIC PROPERTIES OF INTEGRAL CURRENT SPACES

In this section we prove a number of properties of integral current spaces as well as a new Gromov-Hausdorff Compactness Theorem.

3.1. Balls. Many theorems in Riemannian geometry involve balls,

$$(106) \quad B(p, r) = \{x \in X : d_X(x, p) < r\} \quad \bar{B}(p, r) = \{x \in X : d_X(x, p) \leq r\}.$$

Lemma 3.1. A ball in an integral current space, $M = (X, d, T)$, with the current restricted from the current structure of the Riemannian manifold is an integral current space itself,

$$(107) \quad S(p, r) = (\text{set}(T \llcorner B(p, r)), d, T \llcorner B(p, r))$$

for almost every $r > 0$. Furthermore,

$$(108) \quad B(p, r) \subset \text{set}(S(p, r)) \subset \bar{B}(p, r) \subset X.$$

Proof. We first show that $S(p, r) = T \llcorner B(p, r)$ is an integer rectifiable current. Let $\rho_p : \bar{X} \rightarrow \mathbb{R}$ be the distance function from p . Then by Ambrosio-Kirchheim's Slicing Theorem,

$$(109) \quad \partial(T \llcorner B(p, r)) = \partial(T \llcorner \rho_p^{-1}(-\infty, r))$$

$$(110) \quad = \langle T, \rho_p, r \rangle + (\partial T) \llcorner \rho_p^{-1}(-\infty, r)$$

$$(111) \quad = \langle T, \rho_p, r \rangle + (\partial T) \llcorner B(p, r)$$

where the mass of the slice $\langle T, \rho_p, r \rangle$ is bounded for almost every r . Thus

$$(112) \quad \mathbf{M}(\partial(T \llcorner B(p, r))) \leq \mathbf{M}(\langle T, \rho_p, r \rangle) + \mathbf{M}((\partial T) \llcorner B(p, r))$$

$$(113) \quad \leq \mathbf{M}(\langle T, \rho_p, r \rangle) + \mathbf{M}((\partial T) \llcorner B(p, r)) < \infty.$$

So $S(p, r)$ is an integral current in \bar{X} for almost every r .

Next we prove (108). We know $x \in \text{set}(S(p, r)) \subset \bar{X}$ iff

$$(114) \quad 0 < \liminf_{s \rightarrow 0} \frac{\|S(p, r)\|(B(x, s))}{\omega_m s^m}$$

$$(115) \quad = \liminf_{s \rightarrow 0} \frac{\|T\|(B(p, r) \cap B(x, s))}{\omega_m s^m}$$

If $x \in B(p, r) \subset X$, then eventually $B(x, s) \subset B(p, r)$ and the liminf is just the lower density of T at x . Since $x \in X = \text{set}(T)$, this lower density is positive. If $x \in \bar{X} \setminus X$, then the liminf is 0 because it is smaller than the density of T at x , which is 0. If $x \notin \bar{B}(p, r)$, then the liminf is 0 because eventually the balls do not intersect. \square

One may imagine that it is possible that a ball is cusp shaped when we are not in a length space and that some points in the closure of the ball that lie in X do not lie in the set of $S(p, r)$. In a manifold, the set of $S(p, r)$ is a closed ball:

Lemma 3.2. *When M is a Riemannian manifold with boundary*

$$(116) \quad S(p, r) = (\bar{B}(p, r), d, T \llcorner B(p, r))$$

is an integral current space for all $r > 0$.

Proof. In this case,

$$(117) \quad \partial(T \llcorner B(p, r))(f, \pi_1, \dots, \pi_m) = (T \llcorner B(p, r))(1, f, \pi_1, \dots, \pi_m)$$

$$(118) \quad = T(\chi_{B(p, r)}, f, \pi_1, \dots, \pi_m)$$

$$(119) \quad = \int_M \chi_{B(p, r)} df \wedge d\pi_1 \wedge \dots \wedge d\pi_m$$

$$(120) \quad = \int_{B(p, r)} df \wedge d\pi_1 \wedge \dots \wedge d\pi_m$$

$$(121) \quad = \int_{B(p, r)} df \wedge d\pi_1 \wedge \dots \wedge d\pi_m$$

$$(122) \quad = \int_{\partial B(p, r)} f d\pi_1 \wedge \dots \wedge d\pi_m$$

So $\mathbf{M}(\partial(T \llcorner B(p, r))) = \text{Vol}_{m-1}(\partial B(p, r)) < \infty$.

To see that $\bar{B}(p, r) \subset M$ is $\text{set}(S(p, r))$, we refer to (114). If $d(x, p) = r$, then let $\gamma : [0, r] \rightarrow M$ be a curve parametrized by arclength running minimally from x to p . Then

$$(123) \quad B(\gamma(s/2), s/2) \subset B(x, s) \cap B(p, r).$$

and

$$(124) \quad \liminf_{s \rightarrow 0} \frac{\|S(p, r)\|(B(x, s))}{\omega_m s^m} = \liminf_{s \rightarrow 0} \frac{\|T\|(B(p, r) \cap B(x, s))}{\omega_m s^m}$$

$$(125) \quad \geq \liminf_{s \rightarrow 0} \frac{\|T\|(B(\gamma(s/2), s/2))}{\omega_m s^m}$$

$$(126) \quad \geq \liminf_{s \rightarrow 0} \frac{\text{Vol}(B(\gamma(s/2), s/2))}{2^m \omega_m (s/2)^m} \geq \frac{1/2}{2^m}$$

because in a manifold with boundary, we eventually lie within a half plane chart where all tiny balls are either uniformly close to a Euclidean ball or half a Euclidean ball. \square

Example 3.3. *There exist integral current spaces with balls that are not integral current spaces.*

Proof. Suppose we define an integral current space, (X, d, T) where $X = S^2$ with the following generalized metric

$$(127) \quad g = dr^2 + (\cos(r)/r^2)^2 d\theta^2 \quad r \in [-\pi/2, \pi/2].$$

The metric is defined as

$$(128) \quad d(p_1, p_2) = \inf\{L_g(\gamma) : \gamma(0) = p_1, \gamma(1) = p_2\}$$

where

$$(129) \quad L_g(\gamma) = \int_0^1 g(\gamma'(t), \gamma'(t))^{1/2} dt$$

as in a Riemannian manifold. In fact this metric space consists of two open isometric Riemannian manifolds diffeomorphic to disks whose metric completions are glued together along corresponding points. The current structure T is defined by

$$(130) \quad T(f, \pi_1, \dots, \pi_m) = \int_{-\pi/2}^{\pi/2} \int_{S^1} f d\pi_1 \wedge \dots \wedge d\pi_m$$

$$(131) \quad = \int_{-\pi/2}^0 \int_{S^1} f d\pi_1 \wedge \dots \wedge d\pi_m + \int_0^{\pi/2} \int_{S^1} f d\pi_1 \wedge \dots \wedge d\pi_m$$

so that

$$(132) \quad \mathbf{M}(T) = \text{Vol}_m(r^{-1}[-\pi/2, 0)) + \text{Vol}_m(r^{-1}(0, \pi/2]) < \infty$$

and $\partial T = 0$

Setting p such that $r(p) = -\pi/2$ we see that $S(p, \pi/2)$ is a rectifiable current but its boundary does not have finite mass. This can be seen by taking q such that $(r(q), \theta(q)) =$

$(0, 0)$, setting $\pi_1 = \rho_q$ and $f = \rho_p = r + \pi/2$ and observing that

$$(133) \quad |\partial(S(p, \pi/2))(f, \pi_1)| = |S(p, \pi/2)(1, f, \pi_1)|$$

$$(134) \quad = \left| \int_{B(p, \pi/2)} df \wedge d\pi_1 \right|$$

$$(135) \quad \geq \left| \int_{B(p, \pi/2-\delta)} df \wedge d\pi_1 \right|$$

$$(136) \quad = \left| \int_{\partial B(p, \pi/2-\delta)} f d\pi_1 \right|$$

$$(137) \quad = \left| \int_{\theta=-\pi}^{\pi} (\pi/2 - \delta) \frac{d\pi_1}{d\theta} d\theta \right|$$

$$(138) \quad = \left| \int_{\theta=-\pi}^{\pi} (\pi/2 - \delta) \frac{\cos(r)}{r^2} d\theta \right|$$

$$(139) \quad \geq 2\pi(\pi/2 - \delta) \frac{\cos(-\delta)}{\delta^2}$$

which is unbounded as we decrease δ to 0. \square

Remark 3.4. Note that the outside of the ball, $(M \setminus B(p, r), d, T - S(p, r))$, is also an integral current space for almost every $r > 0$.

3.2. Isometric Products. One of the most useful notions in Riemannian geometry is that of an isometric product $M \times I$ of a Riemannian manifold M with an interval, I , endowed with the metric

$$(140) \quad d_{M \times I}((p_1, t_1), (p_2, t_2)) = \sqrt{d_M(p_1, p_2)^2 + |t_1 - t_2|^2}.$$

We need to define the isometric product of an integral current space with an interval:

Definition 3.5. The product of an integral current space, $M^m = (X, d_X, T)$, with an interval I_ϵ , denoted

$$(141) \quad M \times I = (X \times I_\epsilon, d_{X \times I_\epsilon}, T \times I_\epsilon)$$

where $d_{X \times I_\epsilon}$ is defined as in (140) and

$$(142) \quad (T \times I_\epsilon)(f, \pi_1, \dots, \pi_{m+1}) = \sum_{i=1}^{m+1} (-1)^{i+1} \int_0^\epsilon T \left(f_t \frac{\partial \pi_i}{\partial t}, \pi_{1t}, \dots, \hat{\pi}_{it}, \dots, \pi_{(m+1)t} \right) dt$$

where $h_t : \bar{X} \rightarrow \mathbb{R}$ is defined $h_t(x) = h(x, t)$ for any $h : \bar{X} \times I_\epsilon \rightarrow \mathbb{R}$ and where

$$(143) \quad (\pi_{1t}, \dots, \hat{\pi}_{it}, \dots, \pi_{(m+1)t}) = (\pi_{1t}, \pi_{2t}, \dots, \pi_{(i-1)t}, \pi_{(i+1)t}, \dots, \pi_{(m+1)t}).$$

We prove this defines an integral current space in Proposition 3.7 below.

Remark 3.6. This is closely related to the cone construction in Defn 10.1 of [1], however our ambient metric space changes after taking the product and we do not contract to a point. Ambrosio-Kirchheim observe that (142) is well defined because for \mathcal{L}^1 almost every $t \in I_\epsilon$ the partial derivatives are defined for $\|T\|$ almost every $x \in X$. This is also true in our setting. The proof that their cone construction defines a current [1] Theorem 10.2, however, does not extend to our setting because our construction does not close up at a point as theirs does and our construction depends on ϵ but not on the size of a bounding ball.

Proposition 3.7. *Given an integral current space $M = (X, d, T)$, the isometric product $M \times I_\epsilon$ is an integral current space such that*

$$(144) \quad \mathbf{M}(M \times I_\epsilon) = \epsilon \mathbf{M}(M)$$

and such that

$$(145) \quad \partial(T \times I_\epsilon) = (\partial T) \times I_\epsilon + T \times \partial I_\epsilon.$$

where

$$(146) \quad T \times \partial I := \psi_{\epsilon\#} T - \psi_{0\#} T$$

where $\psi_t : \bar{X} \rightarrow \bar{X} \times I_\epsilon$ is the isometric embedding $\psi_t(x) = (x, t)$.

Proof. First we must show $T \times I_\epsilon$ satisfies the three conditions of a current:

Multilinearity follows from the multilinearity of T and the use of the alternating sum in the definition of $T \times I$.

To see locality we suppose there is a π_i which is constant on a neighborhood of $\{f \neq 0\}$. Then $\partial\pi_i/\partial t = 0$ on a neighborhood of $\{f \neq 0\}$ so the i^{th} term in the sum is 0. Since for all $t \in I_\epsilon$, π_{it} is constant on a neighborhood of $\{f_t \neq 0\}$ the rest of the terms are 0 as well by the locality of T .

To prove continuity and finite mass, we will use the fact that T is integer rectifiable. In particular there exists a parametrization as $\varphi_i : A_i \subset \mathbb{R}^m \rightarrow \bar{X}$ and weight functions $\theta_i \in L^1(A_i, \mathbb{N})$ such that

$$(147) \quad T = \sum_{k=1}^{\infty} \varphi_{k\#} [\theta_k].$$

So $(T \times I_\epsilon)(f, \pi_1, \dots, \pi_{m+1}) =$

$$\begin{aligned} &= \sum_{i=1}^{m+1} (-1)^{i+1} \int_0^\epsilon \sum_{k=1}^{\infty} \varphi_{k\#} [\theta_k] \left(f_t \frac{\partial \pi_{it}}{\partial t}, \pi_{1t}, \dots, \hat{\pi}_{it}, \dots, \pi_{(m+1)t} \right) dt \\ &= \sum_{i=1}^{m+1} (-1)^{i+1} \sum_{k=1}^{\infty} \int_{t=0}^\epsilon \int_{A_k} \theta_k f_t \circ \varphi_k \frac{\partial \pi_{it}}{\partial t} \circ \varphi_k d(\pi_{1t} \circ \varphi_k) \wedge \dots \wedge d\hat{\pi}_{it} \wedge \dots \wedge d(\pi_{(m+1)t} \circ \varphi_k) dt \\ &= \sum_{k=1}^{\infty} \int_{A_k} \int_{t=0}^\epsilon \theta_k(x) f(\varphi_k(x), t) \left(\sum_{i=1}^{m+1} (-1)^{i+1} \frac{\partial \pi_{it}}{\partial t} \circ \varphi_k d(\pi_{1t} \circ \varphi_k) \wedge \dots \wedge d\hat{\pi}_{it} \wedge \dots \wedge d(\pi_{(m+1)t} \circ \varphi_k) \right) dt \\ &= \sum_{k=1}^{\infty} \int_{A_k \times I_\epsilon} \theta_k(x) f(\varphi_k(x), t) d(\pi_1 \circ \varphi) \wedge \dots \wedge d(\pi_{m+1} \circ \varphi). \end{aligned}$$

Thus

$$(148) \quad T \times I = \sum_{k=1}^{\infty} \varphi'_{k\#} [\theta'_k].$$

where

$$(149) \quad \varphi'_k : A_k \times I_\epsilon \rightarrow \bar{X} \times I_\epsilon \text{ satisfies } \varphi'_k(x, t) = (\varphi_k(x), t)$$

and $\theta'_k \in L^1(A_k \times I_\epsilon, \mathbb{N})$ satisfies $\theta'_k(x, t) = \theta_k(x)$. Observe that the images of these charts are disjoint and that

$$(150) \quad \mathbf{M}(T \times I_\epsilon) = \sum_{k=1}^{\infty} \mathbf{M}(\varphi'_{k\#}[\theta'_k])$$

$$(151) \quad = \sum_{k=1}^{\infty} \int_{A_k \times I_\epsilon} |\theta'_k| \mathcal{L}^{m+1}$$

$$(152) \quad = \sum_{k=1}^{\infty} \epsilon \int_{A_k} |\theta_k| \mathcal{L}^m$$

$$(153) \quad = \sum_{k=1}^{\infty} \epsilon \mathbf{M}(\varphi_{k\#}[\theta_k]) = \epsilon \mathbf{M}(T).$$

The continuity of $T \times I_\epsilon$ now follows because all integer rectifiable currents defined by parametrizations are currents.

Observe also that if $A \subset \bar{X}$ and $(a_1, a_2) \subset I$ then

$$(154) \quad \|T \times I_\epsilon\|(A \times (a_1, a_2)) = \mathbf{M}((T \times I_\epsilon) \llcorner (A \times (a_1, a_2)))$$

$$(155) \quad = \sum_{k=1}^{\infty} \mathbf{M}((\varphi'_{k\#}[\theta'_k]) \llcorner (A \times (a_1, a_2)))$$

$$(156) \quad = \sum_{k=1}^{\infty} \int_{(A \cap A_k) \times (a_1, a_2)} |\theta'_k| \mathcal{L}^{m+1}$$

$$(157) \quad = \sum_{k=1}^{\infty} (a_2 - a_1) \int_{A \cap A_k} |\theta_k| \mathcal{L}^m$$

$$(158) \quad = \sum_{k=1}^{\infty} (a_2 - a_1) \mathbf{M}(\varphi_{k\#}[\theta_k] \llcorner A)$$

$$(159) \quad = (a_2 - a_1) \mathbf{M}(T \llcorner A) = (a_2 - a_1) \|T\|(A).$$

Thus $\|T \times I_\epsilon\| = \|T\| \times \mathcal{L}^1$

To prove that $T \times I_\epsilon$ is an integral current, we need only verify that the current $\partial(T \times I_\epsilon)$ has finite mass. Applying the Chain Rule [1]Thm 3.5 and Lemma 3.8 (proven below), we have

$$\partial(T \times I_\epsilon)(f, \tau_1, \dots, \tau_m) - ((\partial T) \times I_\epsilon)(f, \tau_1, \dots, \tau_m) =$$

$$\begin{aligned}
&= (T \times I_\epsilon)(1, f, \tau_1, \dots, \tau_m) - \sum_{i=1}^m (-1)^i \int_0^\epsilon \partial T \left(f_t \frac{\partial \tau_i}{\partial t}, \tau_{1t}, \dots, \hat{\tau}_{it}, \dots, \tau_{mt} \right) dt \\
&= \int_0^\epsilon T \left(\frac{\partial f}{\partial t}, \tau_{1t}, \dots, \tau_{mt} \right) dt + \sum_{i=1}^m (-1)^{i+1} \int_0^\epsilon T \left(\frac{\partial \tau_i}{\partial t}, f_t, \tau_{1t}, \dots, \hat{\tau}_{it}, \dots, \tau_{mt} \right) dt \\
&\quad + \sum_{i=1}^m (-1)^{i+1} \int_0^\epsilon T \left(1, f_t \frac{\partial \tau_i}{\partial t}, \tau_{1t}, \dots, \hat{\tau}_{it}, \dots, \tau_{mt} \right) dt \\
&= \int_0^\epsilon T \left(\frac{\partial f}{\partial t}, \tau_{1t}, \dots, \tau_{mt} \right) dt + \sum_{i=1}^m (-1)^i \int_0^\epsilon T \left(f_t, \frac{\partial \tau_i}{\partial t}, \tau_{1t}, \dots, \hat{\tau}_{it}, \dots, \tau_{mt} \right) dt \\
&= \int_0^\epsilon T \left(\frac{\partial f}{\partial t}, \tau_{1t}, \dots, \tau_{mt} \right) dt + \sum_{i=1}^m \int_0^\epsilon T \left(f_t, \tau_{1t}, \dots, \tau_{(i-1)t}, \frac{\partial \tau_i}{\partial t}, \tau_{(i+1)t}, \dots, \tau_{mt} \right) dt \\
&= \int_0^\epsilon \frac{\partial}{\partial t} T(f_t, \tau_{1t}, \dots, \tau_{mt}) dt \\
&= T(f_\epsilon, \tau_{1\epsilon}, \dots, \tau_{m\epsilon}) - T(f_0, \tau_{10}, \dots, \tau_{m0}) \\
&= \psi_{\epsilon\#} T(f, \tau_1, \dots, \tau_m) - \psi_{0\#} T(f, \tau_1, \dots, \tau_m) \\
&= T \times \partial I(f, \tau_1, \dots, \tau_m).
\end{aligned}$$

Thus we have (145).

Observe that $T \times \partial I$ is an integral current because it is the sum of push forwards of integral currents and that

$$(160) \quad \mathbf{M}(T \times \partial I_\epsilon) = 2\mathbf{M}(T).$$

Since we know products are rectifiable, $(\partial T) \times I_\epsilon$ is rectifiable and has finite mass $\leq \epsilon \mathbf{M}(\partial T)$. Thus applying (145) we see that

$$(161) \quad \mathbf{M}(\partial(T \times I_\epsilon)) \leq \mathbf{M}((\partial T) \times I_\epsilon) + \mathbf{M}(T \times \partial I_\epsilon) \leq \epsilon \mathbf{M}(\partial T) + 2\mathbf{M}(T).$$

Thus the current structure of $M \times I_\epsilon$ is an integral current.

Lastly we verify that

$$(162) \quad \text{set}(T \times I_\epsilon) = \text{set}(T) \times I_\epsilon.$$

Given $(p, t) \in \bar{X} \times I_\epsilon$, then the following statements are equivalent:

$$(163) \quad (p, t) \in \text{set}(T \times I_\epsilon).$$

$$(164) \quad 0 < \liminf_{r \rightarrow 0} \frac{\|T \times I_\epsilon\|(B_{(p,t)}(r))}{r^{m+1}}.$$

$$(165) \quad 0 < \liminf_{r \rightarrow 0} \frac{\|T \times I_\epsilon\|(B_p(r) \times (t-r, t+r))}{r^{m+1}}.$$

$$(166) \quad 0 < \liminf_{r \rightarrow 0} \frac{2r\|T\|(B_p(r))}{r^{m+1}}.$$

$$(167) \quad 0 < \liminf_{r \rightarrow 0} \frac{\|T\|(B_p(r))}{r^m}.$$

$$(168) \quad p \in \text{set}(T).$$

The proposition follows. \square

Lemma 3.8. *If π_{it} are Lipschitz in $Z \times I$, and $T \in \mathbf{I}_m(Z)$ then for almost every $t \in I$,*

$$(169) \quad \frac{\partial}{\partial t} T(\pi_{0t}, \dots, \pi_{mt}) = \sum_{i=0}^m T\left(\pi_{0t}, \dots, \pi_{(i-1)t}, \frac{\partial \pi_i}{\partial t}, \pi_{(i+1)t}, \dots, \pi_{mt}\right)$$

Proof. This follows from the multilinearity of T , the usual expansion of the difference quotient as a sum of difference quotients in which one term changes at a time, the fact the T is continuous with respect to pointwise convergence and that the difference quotients have pointwise limits for almost every $t \in I$ because the π_i are Lipschitz in t . \square

Proposition added in August 2012:

The following proposition will be applied later when studying limits under intrinsic flat and Gromov-Hausdorff convergence.

Proposition 3.9. *Suppose M_i^m are integral current spaces and $\epsilon > 0$, then*

$$(170) \quad d_{GH}(M_1^m \times I_\epsilon, M_2^m \times I_\epsilon) \leq d_{GH}(M_1^m, M_2^m),$$

and

$$(171) \quad d_{\mathcal{F}}(M_1^m \times I_\epsilon, M_2^m \times I_\epsilon) \leq (2 + \epsilon)d_{\mathcal{F}}(M_1^m, M_2^m).$$

Proof. Since the intrinsic flat distance is achieved, there exists a metric space Z and isometric embeddings $\varphi_i : M_i \rightarrow Z$, and integral currents A, B on Z such that

$$(172) \quad \varphi_{1\#}M_1 - \varphi_{2\#}M_2 = A + \partial B$$

and

$$(173) \quad d_{\mathcal{F}}(M_1^m, M_2^m) = (\mathbf{M}(A) + \mathbf{M}(B)).$$

Setting $Z' = Z \times I_\epsilon$ endowed with the product metric, we have isometric embeddings $\varphi'_i : M_i \times I_\epsilon \rightarrow Z'$ and we have integral currents $A' = A \times I_\epsilon$ and $B' = B \times I_\epsilon$ such that

$$(174) \quad \varphi'_{1\#}(M_1 \times I_\epsilon) - \varphi'_{2\#}(M_2 \times I_\epsilon) = (\varphi'_{1\#}M_1) \times I_\epsilon - (\varphi_{2\#}M_2) \times I_\epsilon$$

$$(175) \quad = (\varphi'_{1\#}M_1) - \varphi_{2\#}M_2 \times I_\epsilon$$

$$(176) \quad = (A + \partial B) \times I_\epsilon$$

$$(177) \quad = A \times I_\epsilon + \partial(B \times I_\epsilon) - B \times (\partial I_\epsilon).$$

Thus by Proposition 3.7 and (160) we have

$$(178) \quad d_{\mathcal{F}}(M_1^m \times I_\epsilon, M_2^m \times I_\epsilon) \leq \mathbf{M}(A \times I_\epsilon) + \mathbf{M}(B \times I_\epsilon) + \mathbf{M}(B \times (\partial I_\epsilon))$$

$$(179) \quad \leq \epsilon \mathbf{M}(A) + \epsilon \mathbf{M}(B) + 2\mathbf{M}(B)$$

$$(180) \quad \leq (2 + \epsilon)\mathbf{M}(A) + (2 + \epsilon)\mathbf{M}(B)$$

$$(181) \quad = 3d_{\mathcal{F}}(M_1^m, M_2^m).$$

To see the Gromov-Hausdorff estimate, one needs only observe that whenever $Y_1 \subset T_r(Y_2) \subset Z$, then

$$(182) \quad Y_1 \times I_\epsilon \subset T_r(Y_2 \times I_\epsilon) \subset Z \times I_\epsilon.$$

\square

3.3. Slices and Spheres. While balls are a very natural object in metric spaces, a more important notion in integral current spaces is that of a slice. The following proposition follows immediately from the Ambrosio-Kirchheim Slicing Theorem (c.f. Theorem 2.23 and Remark 2.25):

Proposition 3.10. *Given an m dimensional integral current space $M = (X, d, T)$ and Lipschitz functions $F : X \rightarrow \mathbb{R}^k$ where $k < m$, then for almost every $t \in \mathbb{R}^k$, we can define an $m - k$ dimensional integral current space called the slice of (X, d, T) :*

$$(183) \quad \text{Slice}(M, F, t) = \text{Slice}(F, t) = (\text{set} < T, F, t >, d, < T, F, t >)$$

where $< T, F, t > = < T, F_1, \dots, F_k, t_1, \dots, t_k >$ is an integral current on \bar{X} defined using the Ambrosio-Kirchheim Slicing Theorem and $\text{set} < T, F, t > \subset F^{-1}(t)$. We can integrate the masses of slices to obtain lower bounds of the mass of the original space:

$$(184) \quad \int_{t \in \mathbb{R}^k} \mathbf{M}(\text{Slice}(M, F, t)) \mathcal{L}^k \leq \prod_{j=1}^k \text{Lip}(F_j) \mathbf{M}(T).$$

and $\partial \text{Slice}(M, F, t) = (-1)^k \text{Slice}(\partial M, F, t)$.

Proof. This proposition follows immediately from the Ambrosio-Kirchheim Slicing Theorem 5.6 using the fact that F has a unique extension to \bar{X} and Defn 2.5. The last part follows from Lemma 5.8. \square

Example 3.11. In Example 3.3, where $\partial T = \emptyset$ we see that taking $F : X \rightarrow \mathbb{R}$ to be the distance function from p the slice $\text{Slice}(M, f, s) = \partial S(p, s)$ is an integral current space only when $s \neq \pi/2$.

Lemma 3.12. *Given an m dimensional integral current space (X, d, T) and a point p then for almost every $r \in \mathbb{R}$, we can define an $m - 1$ dimensional integral current space called the sphere about p of radius r :*

$$(185) \quad \text{Sphere}(p, r) = \text{Slice}(\rho_p, r)$$

On a Riemannian manifold with boundary,

$$(186) \quad \text{Sphere}(p, r)(f, \pi_1, \dots, \pi_{m-1}) = \int_{\rho_p^{-1}(r)} f d\pi_1 \wedge \dots \wedge d\pi_{m-1}$$

is an integral current space for all $r \in \mathbb{R}$.

Proof. This follows from Proposition 3.10 and the Ambrosio Kirchheim Slicing Theorem (c.f. Theorem 2.23 and the fact that $\text{Lip}(\rho_p) = 1$. The Riemannian part follows from Stoke's Theorem and the fact that spheres of all radii in Riemannian manifolds have finite volume as can be seen either by applying the Ricatti equation or Jacobi fields. \square

Observe the distinction between the sphere and the boundary of a ball in Lemma 3.12 when M has boundary. Next we examine the setting when we do not hit the boundary:

Lemma 3.13. *When $\text{set}(\partial T) \cap \bar{B}(p, R) \subset \bar{X}$ is empty then*

$$(187) \quad \text{Sphere}(p, R) = \partial S(p, R).$$

Furthermore,

$$(188) \quad \int_0^R \mathbf{M}(\partial S(p, r)) d\mathcal{L}(r) \leq \mathbf{M}(S(p, R)).$$

In particular, on an open Riemannian manifold, for any $p \in M$, there is a sufficiently small $R > 0$ such that this lemma holds. On a Riemannian manifold without boundary, these hold for all $R > 0$.

Proof. This follows from Proposition 3.10 and Theorem 2.23. \square

Lemma 3.14. *Given an m dimensional integral current space (X, d, T) and a $\rho : X \rightarrow \mathbb{R}$ a Lipschitz function with $\text{Lip}(\rho) \leq 1$ then for almost every $r \in \mathbb{R}$, we can define an $m - 1$ dimensional integral current space, $\text{Slice}(\rho, r)$ where*

$$(189) \quad \int_{-\infty}^{\infty} \mathbf{M}(\text{Slice}(\rho, r)) \, d\mathcal{L}(r) \leq \mathbf{M}(T).$$

On a Riemannian manifold with boundary

$$(190) \quad \text{Slice}(\rho, r)(f, \pi_1, \dots, \pi_{m-1}) = \int_{\rho^{-1}(r)} f \, d\pi_1 \wedge \dots \wedge d\pi_{m-1}$$

is defined for all $r \in \mathbb{R}$.

Proof. This follows from Proposition 3.10 and Theorem 2.23. \square

Lemma 3.15. *Given an m dimensional integral current space (X, d, T) and a $\rho : X \rightarrow \mathbb{R}^k$ have $\text{Lip}(\rho_i) \leq 1$ then for almost every $r \in \mathbb{R}^k$, we can define an $m - k$ dimensional integral current space, $\text{Slice}(\rho, r)$ where*

$$(191) \quad \int_{\mathbb{R}^k} \mathbf{M}(\text{Slice}(\rho, r)) \, d\mathcal{L}(r) \leq \mathbf{M}(T).$$

Proof. This follows from Proposition 3.10 and Theorem 2.23. \square

Remark 3.16. *On a Riemannian manifold with boundary*

$$(192) \quad \text{Slice}(\rho, r)(f, \pi_1, \dots, \pi_{m-1}) = \int_{\rho^{-1}(r)} f \, d\pi_1 \wedge \dots \wedge d\pi_{m-1}$$

is defined for all $r \in \mathbb{R}$ such that $\rho_p^{-1}(r)$ is $m - 1$ dimensional. By the above lemma this will be true for almost every r . Note, however, that if ρ_i are distance functions from poorly chosen points, the slice may be the 0 space for almost every r because $\rho_p^{-1}(r) = \emptyset$. This occurs for example on the standard three dimensional sphere if we take ρ_1, ρ_2 to be distance functions from opposite poles.

3.4. Filling Volumes of Spheres and Slices. The following lemmas were applied without proof in [19]. We may now easily prove them.

Lemma 3.17. *Given an integral current space, $M^m = (X, d, T)$,*

$$(193) \quad \mathbf{M}(S(p, r)) \geq \text{FillVol}(\partial S(p, r)) \quad \forall p \in \bar{X}.$$

Thus $p \in \bar{X}$ lies in $X = \text{set}(T)$ iff

$$(194) \quad \liminf_{r \rightarrow 0} \text{FillVol}(\partial S(p, r))/r^m > 0.$$

Proof. This follows immediately from the definition of filling volume and the definition of $\text{set}(T)$. \square

Lemma 3.18. *Given an integral current space, $M = (X, d, T)$. If $B_p(r) \cap \partial M = \emptyset$ then*

$$(195) \quad \mathbf{M}(S(p, r)) \geq \text{FillVol}(\text{Sphere}(p, R)).$$

Thus if $\partial M = \emptyset$, we know that $p \in \bar{X}$ lies in $X = \text{set}(T)$ iff

$$(196) \quad \liminf_{r \rightarrow 0} \text{FillVol}(\text{Sphere}(p, r))/r^m > 0.$$

Proof. This follows immediately from Lemma 3.13 and Lemma 3.17. \square

Theorem 4.1 of [19] can be stated as follows:

Theorem 3.19 (Sormani-Wenger). *Suppose $M^m = (X, d, T)$ is a compact Riemannian manifold such that there exists $r_0 > 0, k > 0$ such that $\bar{B}(p, kr_0) \cap \partial M = \emptyset$ and every $B(x, r) \subset \bar{B}(p, r_0)$ is contractible within $B(x, kr) \subset \bar{B}(p, r_0)$ then $\exists C_k$ such that*

$$(197) \quad \text{Vol}(\bar{B}(x, r)) = \|T\|(\bar{B}(x, r)) \geq \text{FillVol}(\partial S(p, r)) \geq C_k r^m.$$

This theorem essentially follows from a result of Greene-Petersen [7] combined with Lemma 3.18. The statement in [20] applies to a more general class of spaces and requires a much more subtle proof involving Lipschitz extensions.

3.5. Sliced Filling Volumes of Balls. Spheres aren't the only slices whose filling volumes may be used to estimate the volumes of balls. We define the following new notions:

Definition 3.20. *Given an integral current space, $M^m = (X, d, T)$ and $F_1, F_2, \dots, F_k : M \rightarrow \mathbb{R}$ with $k \leq m - 1$ are Lipschitz functions with Lipschitz constant $\text{Lip}(F_j) = \lambda_j$ then we define the sliced filling volume of $\partial S(p, r) \in \mathbf{I}_{m-1}(\bar{X})$, to be*

$$(198) \quad \mathbf{SF}(p, r, F_1, \dots, F_k) = \int_{t \in A_r} \text{FillVol}(\partial \text{Slice}(S(p, r), F, t)) \mathcal{L}^k.$$

where

$$(199) \quad A_r = [\min F_1, \max F_1] \times [\min F_2, \max F_2] \times \dots \times [\min F_k, \max F_k]$$

where $\min F_j = \min\{F_j(x) : x \in \bar{B}_p(r)\}$ and $\max F_j = \max\{F_j(x) : x \in \bar{B}_p(r)\}$. Given $q_1, \dots, q_k \in M$, we set,

$$(200) \quad \mathbf{SF}(p, r, q_1, \dots, q_k) = \mathbf{SF}(p, r, \rho_1, \dots, \rho_k) \text{ where } \rho_i(x) = d_X(q_i, x).$$

Lemma 3.21. *Given an integral current space, $M^m = (X, d, T)$ and $F_1, F_2, \dots, F_k : M \rightarrow \mathbb{R}$ with $k \leq m - 1$ are Lipschitz functions with Lipschitz constants, $\text{Lip}(F_j) = \lambda_j > 0$, then*

$$(201) \quad \mathbf{M}(S(p, r)) \geq \prod_{j=1}^k \lambda_j^{-1} \mathbf{SF}(p, r, F_1, \dots, F_k).$$

Thus $p \in \bar{X}$ lies in $X = \text{set}(T)$ if

$$(202) \quad \liminf_{r \rightarrow 0} \frac{1}{r^m} \mathbf{SF}(p, r, F_1, \dots, F_k) > 0.$$

Proof. By Proposition 3.10 we know

$$(203) \quad \mathbf{M}(S(p, r)) \geq \prod_{j=1}^k \lambda_j^{-1} \mathbf{M}(S(p, r) \llcorner dF)$$

$$(204) \quad \geq \prod_{j=1}^k \lambda_j^{-1} \int_{t \in \mathbb{R}^k} \mathbf{M}(\text{Slice}(S(p, r), F, t)) \mathcal{L}^k$$

$$(205) \quad = \prod_{j=1}^k \lambda_j^{-1} \int_{t \in A} \mathbf{M}(\text{Slice}(S(p, r), F, t)) \mathcal{L}^k.$$

The lemma then follows from $k \leq m - 1$ implies each slice is at least 1 dimensional and then applying (91) and the fact that $\partial < S, F, t > = - < \partial S, F, t >$. \square

3.6. Uniform \mathbf{SF}_k .

Definition 3.22. Given an integral current space M^m and $p \in M^m$, then for almost every r , we can define the k^{th} sliced filling,

$$(206) \quad \mathbf{SF}_k(p, r) = \sup\{\mathbf{SF}(p, r, q_1, \dots, q_k) : q_i \in \partial B_p(r)\}.$$

Theorem 3.23. If $M_i^m = (X_i, d_i, T_i)$ are integral current spaces with a uniform upper bound on $\text{Vol}(M_i) \leq V_0$ and diameter $\text{Diam}(M_i) \leq D_0$, and a uniform lower bound on the k^{th} sliced filling

$$(207) \quad \mathbf{SF}_k(p, r) \geq C(r) > 0$$

for all $p \in M_i$, for all i and for almost every $r \in \mathbb{R}$, then a subsequence (X_i, d_i) converges in the Gromov-Hausdorff sense to a limit space (Y, d_Y) .

Later we will prove that the subsequence converges in the intrinsic flat sense to the same limit space when $C(r) \geq C_{SF} r^m > 0$.

Proof. For any p in any M_i , there exist q_1, \dots, q_k such that

$$(208) \quad \mathbf{SF}(p, r, q_1, \dots, q_k) \geq C(r)/2 > 0.$$

So $\mathbf{M}(S(p, r)) \geq C(r)/2$. Thus the number of disjoint balls of radius r in M_i is $\leq 2V_0/C(r)$. So we may apply Gromov's Compactness Theorem. \square

3.7. Filling Volumes of 0 Dimensional Spaces. Before proceeding we need the following lemma:

Lemma 3.24. Let M be an integral current space. Suppose $S \in \mathbf{I}_1(M)$ such that $\partial S \neq \mathbf{0}$. Then $\text{set}(\partial S) = \{p_1, \dots, p_N\}$ with $N \geq 2$ and

$$(209) \quad \partial S(f) = \sum_{i=1}^N \sigma_i \theta_i f(p_i)$$

where $\theta_i \in \mathbb{Z}^+$ and $|\sigma_i| = 1$ and

$$(210) \quad \text{FillVol}(\partial S) \geq \max_{j=1..N} \left(|\theta_j| \min_{i \neq j} d_X(p_i, p_j) \right) > 0$$

In particular

$$(211) \quad \text{FillVol}(\partial S) \geq \inf \{d_X(p_i, p_j) : i, j \in \{1, 2, \dots, N\}\} \geq \inf \{d(x, y) : x \neq y, x, y \in \text{set}(\partial S)\} > 0.$$

Proof. Recall that by Remark 2.34, ∂S satisfies (209) where $\sum_{i=1}^N \sigma_i \theta_i = 0$. So $N \geq 2$ when $\partial S \neq \mathbf{0}$.

Suppose $M' = (Y, d_Y, T)$ is any one dimensional integral current space with a current preserving isometry $\varphi : \text{set}(\partial M') \rightarrow \text{set}(\partial S) \subset \bar{X}$ so that

$$(212) \quad \varphi_{\#} \partial T = \partial S \in \mathbf{I}_0(M)$$

and $d_X(\varphi(y_1), \varphi(y_2)) = d_Y(y_1, y_2)$ for all $y_1, y_2 \in \text{set}(T) \subset Y$. In particular there exist distinct points

$$(213) \quad p'_j = \varphi^{-1}(p_j) \in \bar{Y}$$

such that for any Lipschitz $f : \bar{Y} \rightarrow \mathbb{R}$ we have

$$(214) \quad T(1, f) = \partial(T)(f) = \sum_{i=1}^N \sigma_i \theta_i f(p'_i).$$

By (12) we have

$$(215) \quad |T(1, f)| \leq \text{Lip}(f) \mathbf{M}(T).$$

Let $f_j(y) = \min_{i \neq j} d_Y(y, p'_i)$. Then we have, $\text{Lip}(f_j) = 1$, so

$$(216) \quad \mathbf{M}(T) \geq \left| \sum_{i=1}^N \sigma_i \theta_i f_j(p'_i) \right|$$

$$(217) \quad \geq \theta_j f_j(p'_j) = \theta_j \min_{i \neq j} d_Y(p'_i, p'_j)$$

$$(218) \quad = \theta_j \min_{i \neq j} d_X(p_i, p_j)$$

Taking an infimum over all T , we have,

$$(219) \quad \text{FillVol}(\partial S) \geq \theta_j \min_{i \neq j} d_X(p_i, p_j).$$

As this is true for all $j = 1..N$, we have (210). Since $\theta_j \in \mathbb{Z}^+$, we have the simpler lower bound given in (211). \square

3.8. Masses of Balls from Distances. Here we provide a lower bound on the mass of a ball using a sliced filling volume and estimates on the filling volumes of 0 dimensional currents.

Theorem 3.25. *Given an integral current space, $M^m = (X, d, T)$ and points $p_1, \dots, p_{m-1} \in X$, then then if $\bar{B}_p(r) \cap \text{set}(\partial T) = \emptyset$ we have for almost every $r > 0$,*

$$(220) \quad \mathbf{M}(S(p, r)) \geq \mathbf{SF}(p, r, p_1, \dots, p_{m-1}) \geq \int_{s_1-r}^{s_1+r} \cdots \int_{s_{m-1}-r}^{s_{m-1}+r} h(p, r, t_1, \dots, t_{m-1}) dt_1 dt_2 \dots dt_{m-1}$$

where $s_i = d(p_i, p_0)$ and

$$(221) \quad h(p, r, t_1, \dots, t_{m-1}) = \inf \{d(x, y) : x \neq y, x, y \in P(p, r, t_1, \dots, t_{m-1})\}$$

when $P(p, r, t_1, \dots, t_{m-1}) = \bar{B}_p(r) \cap \rho_{p_1}^{-1}(t_1) \cap \cdots \cap \rho_{p_{m-1}}^{-1}(t_{m-1})$ is a nonempty discrete set of points and 0 otherwise.

Thus $p \in \bar{X} \setminus \text{Cl}(\text{set}(\partial T))$ lies in $X = \text{set}(T)$ if

$$(222) \quad \liminf_{r \rightarrow 0} (1/r^m) \int_{t_1=s_1-r}^{s_1+r} \cdots \int_{t_{m-1}=s_{m-1}-r}^{s_{m-1}+r} h(p, r, t_1, \dots, t_{m-1}) dt_1 dt_2 \dots dt_{m-1} > 0$$

Theorem 3.25 is in fact a special case of the following theorem:

Theorem 3.26. *Given an integral current space, $M^m = (X, d, T)$ and $F_1, F_2, \dots, F_{m-1} : M \rightarrow \mathbb{R}$ are Lipschitz functions with Lipschitz constants, $\text{Lip}(F_j) = \lambda_j$, then for almost every $r > 0$*

$$(223) \quad \mathbf{M}(S(p, r)) \geq \mathbf{SF}(p, r, F_1, \dots, F_k) \geq \prod_{j=1}^k \lambda_j^{-1} \int_{t \in A_r} h(p, r, F, t) d\mathcal{L}^k$$

where

$$(224) \quad h(p, r, F, t) = \inf\{d(x, y) : x \neq y, x, y \in \text{set}(\partial \text{Slice}(S(p, r), F, t))\} > 0$$

when $\partial \text{Slice}(S(p, r), F, t) \in \mathbf{I}_0(\bar{X}) \setminus \{\emptyset\}$ and $h(p, r, F, t) = 0$ otherwise, and where

$$(225) \quad A_r = [\min F_1, \max F_1] \times [\min F_2, \max F_2] \times \dots \times [\min F_k, \max F_k]$$

with $\min F_j = \min\{F_j(x) : x \in \bar{B}_p(r)\}$ and $\max F_j = \max\{F_j(x) : x \in \bar{B}_p(r)\}$.

Before presenting the proof we give two important examples:

Example 3.27. *On Euclidean space, \mathbb{E}^m , taking $F_i : \mathbb{E}^m \rightarrow \mathbb{R}$ to be a collection of perpendicular coordinate functions for $i = 1..m$, $F_i(x_1, \dots, x_m) = x_i$, we have $\lambda_i = 1$ and*

$$(226) \quad h(p, r, F_1, \dots, F_{m-1}, t_1, \dots, t_{m-1}) = 2 \sqrt{r^2 - (t_1^2 + \dots + t_{m-1}^2)}.$$

So

$$(227) \quad \omega_m r^m = \mathbf{M}(S(p, r)) \geq \mathbf{SF}(p, r, F_1, \dots, F_k) = \omega_m r^m.$$

Example 3.28. *On the standard sphere, S^2 , taking $p_1 \in \partial B_p(\pi/2)$ and $r = \pi/2$ and $F_1(x) = d(p_1, x)$, then*

$$(228) \quad h(p, \pi/2, F_1, t) = \min\{2t, 2(\pi - t)\}$$

because the distances are shortest if one travels within the great circle, $\partial B_p(\pi/2)$. So

$$(229) \quad 2\pi = \text{Vol}(S^2_+) = \mathbf{M}(S(p, \pi/2)) \geq \mathbf{SF}(p, \pi/2, F_1)$$

with

$$(230) \quad \mathbf{SF}(p, \pi/2, F_1) = \int_0^\pi h(p, \pi/2, F_1, t) dt$$

$$(231) \quad = 2 \int_0^{\pi/2} 2t dt = 2(\pi/2)^2 = \pi^2/2.$$

Proof. Theorem 3.25 follows from Theorem 3.26 taking $F(x) = (F_1(x), \dots, F_{m-1}(x))$ where $F_i(x) = \rho_{p_i}(x)$. When $\bar{B}(p, r) \cap \text{set} \partial T = \emptyset$, then for almost every $r \in \mathbb{R}$, $t_1 \in \mathbb{R}, \dots, t_{m-1} \in \mathbb{R}$ $\partial \text{Slice}(S(p, r), F, t) \in \mathbf{I}_0(\bar{X})$ and

$$(232) \quad \text{set}(\partial \text{Slice}(S(p, r), F, t)) = \rho_p^{-1}(r) \cap F_1^{-1}(t_1) \cap \dots \cap F_{m-1}^{-1}(t_{m-1}).$$

so this set either has 0 points or at least two points. \square

Proof. Theorem 3.26 is proven by applying Lemma 3.21 to F and then computing the filling volume of the 0 dimensional current, $\partial(\text{Slice}(S(p, r), F, t))$, using Lemma 3.24 stated and proven below. Observe that if $t_i < s_i - r$ or $t_i > s_i + r$ then $h(p, r, t_1, \dots, t_{m-1}) = 0$ because $\rho_p^{-1}(r) \cap \rho_{p_i}^{-1}(t_i) = \emptyset$. \square

Remark 3.29. *Naturally we could combine Theorem 3.26 with any other lower bound on the filling volumes of 0 dimensional sets, like, for example, (210).*

3.9. Tetrahedral Property. Theorem 3.25 allows us to estimate the masses of balls using a tetrahedral property (see Figure 1).

Definition 3.30. Given $C > 0$ and $\beta \in (0, 1)$, a metric space X is said to have the m dimensional C, β -tetrahedral property at a point p for radius r if one can find points $p_1, \dots, p_{m-1} \subset \partial B_p(r) \subset \bar{X}$, such that

$$(233) \quad h(p, r, t_1, \dots, t_{m-1}) \geq Cr \quad \forall (t_1, \dots, t_{m-1}) \in [(1 - \beta)r, (1 + \beta)r]^m$$

where

$$(234) \quad h(p, r, t_1, \dots, t_{m-1}) = \inf \{d(x, y) : x \neq y, x, y \in P(p, r, t_1, \dots, t_{m-1})\}$$

when

$$(235) \quad P(p, r, t_1, \dots, t_{m-1}) = \rho_p^{-1}(r) \cap \rho_{p_1}^{-1}(t_1) \cap \dots \cap \rho_{p_{m-1}}^{-1}(t_{m-1})$$

is nonempty and 0 otherwise. In particular $P(p, r, t_1, \dots, t_{m-1})$ is a discrete set of points.

Example 3.31. On Euclidean space, \mathbb{E}^3 , taking $p_1, p_2 \in \partial B(p, r)$ to such that $d(p_1, p_2) = r$, then there exists exactly two points $x, y \in P(p, r, r, r)$ each forming a tetrahedron with p, p_1, p_2 . See Figure 1. As we vary $t_1, t_2 \in (r/2, 3r/2)$, we still have exactly two points in $P(p, r, t_1, t_2)$. By scaling we see that

$$(236) \quad h(p, r, t_1, t_2) = rh(p, 1, t_1/r, t_2/r) \geq C_{\mathbb{E}^3} r$$

where

$$(237) \quad C_{\mathbb{E}^3} = \inf \{h(p, 1, s_1, s_2) : s_i \in (1/2, 3/2)\} > 0$$

could be computed explicitly. So \mathbb{E}^3 satisfies the $C_{\mathbb{E}^3}, (1/2)$ tetrahedral property at p for all r .

Example 3.32. On a torus, $M_\epsilon^3 = S^1 \times S^1 \times S_\epsilon^1$ where S_ϵ^1 has been scaled to have diameter ϵ instead of π , we see that M^3 satisfies the $C_{\mathbb{E}^3}, (1/2)$ tetrahedral property at p for all $r < \epsilon/4$. By taking $r < \epsilon/4$, we guarantee that the shortest paths between x and y stay within the ball $B(p, r)$ allowing us to use the Euclidean estimates. If r is too large, $P(p, r, t_1, t_2) = \emptyset$.

Remark 3.33. On a Riemannian manifold or an integral current space, we know that $P(p, r, t_1, \dots, t_{m-1})$ is the set of a 0 current which is a boundary. So if it is not empty, it has at least two points, one with positive weight and one with negative weight.

Remark 3.34. It is not just a simple application of the triangle inequality to proceed from knowing $h(p, r, r, \dots, r) \geq Cr$ to having $h(p, r, t_1, \dots, t_{m-1}) \geq C_2 r$. There is the possibility that $P(p, r, t_1, \dots, t_{m-1})$ is empty or has a closest pair of points both near a single point of $C(p, r, r, \dots, r)$ even in a Riemannian manifold. However one expects the same type of curvature conditions that would lead to control of $h(p, r, \dots, r)$ could be used to study $h(p, t_1, \dots, t_{m-1})$.

Remark 3.35. On a manifold with sectional curvature bounded below, one should have the $C, 1/2$ tetrahedral property at any point p as long as $r < \text{injr}(p)/4$ where C depends on the lower sectional curvature bound. This should be provable using the Toponogov Comparison Theorem. One would like to replace the condition on injectivity radius with radius depending upon a lower bound on volume. Work in this direction is under preparation by the author's doctoral students. Note that there is no uniform tetrahedral property on manifolds with positive scalar curvature even when the volume of the balls are uniformly bounded below by that of Euclidean balls [Remark 4.32]. With lower bounds on Ricci curvature one might expect to have the $C, 1/2$ tetrahedral property or an integral version

of this property. Again a uniform lower bound on volume will be necessary as seen in the torus example above.

3.10. Integral Tetrahedral Property. For our applications we need only the following property which clearly holds at any point with the tetrahedral property:

Definition 3.36. Given $C > 0$ and $\beta \in (0, 1)$, a metric space X is said to have the m dimensional integral C, β -tetrahedral property at a point p for radius r if one can find points $p_1, \dots, p_{m-1} \subset \partial B_p(r) \subset \bar{X}$, such that

$$(238) \quad \int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} h(p, r, t_1, \dots, t_{m-1}) dt_1 dt_2 \dots dt_{m-1} \geq C(2\beta)^{m-1} r^m.$$

Proposition 3.37. If X is a metric space that satisfies the $C\beta$ tetrahedral property at p for radius r then it has the $C\beta$ integral tetrahedral property.

Proof.

$$\begin{aligned} \int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} h(p, r, t_1, \dots, t_{m-1}) dt_1 dt_2 \dots dt_{m-1} &\geq \\ &\geq \int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} CR dt_1 dt_2 \dots dt_{m-1} \\ &\geq CR ((1+\beta)r - (1-\beta)r)^{m-1} \end{aligned}$$

□

3.11. Tetrahedral Property and Masses of Balls.

Theorem 3.38. If (X, d, T) is an integral current space and \bar{X} has the m dimensional (integral) C, β -tetrahedral property at a point p for radius r such that $\bar{B}_p(r) \cap \text{set} \partial T = \emptyset$ then

$$(239) \quad \mathbf{M}(S(p, r)) \geq \mathbf{SF}_{m-1}(p, r) \geq C(2\beta)^{m-1} r^m$$

Proof. By Theorem 3.25 with $s_i = r$ we have

$$\begin{aligned} \mathbf{M}(S(p, r)) &\geq \mathbf{SF}(p, r, p_1, \dots, p_{m-1}) \\ &\geq \int_{t_1=s_1-r}^{s_1+r} \cdots \int_{t_{m-1}=s_{m-1}-r}^{s_{m-1}+r} h(P_{(r, t_1, \dots, t_{m-1})}) dt_1 dt_2 \dots dt_{m-1} \\ &\geq \mathbf{SF}(p, r, p_1, \dots, p_{m-1}) \\ &\geq \int_{t_1=0}^{2r} \cdots \int_{t_{m-1}=0}^{2r} h(P_{(r, t_1, \dots, t_{m-1})}) dt_1 dt_2 \dots dt_{m-1} \\ &> \int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} h(p, r, t_1, \dots, t_{m-1}) dt_1 dt_2 \dots dt_{m-1} \\ &> C(2\beta)^{m-1} r^m \end{aligned}$$

□

Theorem 3.39. If p_0 lies in a Riemannian manifold with boundary and $B_{p_0}(r) \cap \partial M = \emptyset$, that has the m dimensional (integral) C, β -tetrahedral property at a point p for radius R such that $\bar{B}_p(r) \cap \text{set} \partial T = \emptyset$ then

$$(240) \quad \text{Vol}(B(p, r)) \geq C(2\beta)^{m-1} r^m$$

Proof. This is an immediate consequence of Theorem 3.38.

□

Remark 3.40. In Example 3.32, as $\epsilon \rightarrow 0$, the $\text{Vol}(B(p, r)) \leq \text{Vol}(M_\epsilon^3) \rightarrow 0$, so we could not have a uniform tetrahedral property on these spaces.

Theorem 3.41. Given $r_0 > 0, \beta \in (0, 1), C > 0, V_0 > 0$. If a sequence of compact Riemannian manifolds, M^m , has $\text{Vol}(M^m) \leq V_0$, $\text{Diam}(M^m) \leq D_0$, and the C, β (integral) tetrahedral property for all balls of radius $\leq r_0$, then a subsequence converges in the Gromov-Hausdorff sense. In particular they have a uniform upper bound on diameter depending only on these constants.

The proof of this theorem strongly requires that the manifold have no boundary.

Proof. This follows immediately from Theorem 3.39 and Gromov's Compactness Theorem, using the fact that we can bound the number of disjoint balls of radius $\epsilon > 0$ in M^m . In a manifold, this provides an upper bound on the diameter of M^m , \square

Later we will apply the following theorem to prove that the Gromov-Hausdorff limit is in fact an Intrinsic Flat limit and thus is countably \mathcal{H}^m rectifiable [Theorem 4.31]. As it does not require a uniform tetrahedral property, it will also be applied in other settings where one has no Gromov-Hausdorff limit [Theorem ??] TODO.

Theorem 3.42. Given an integral current space (X, d, T) and a point $p_0 \in \bar{X} \setminus \text{Cl}(\text{set}(\partial T))$ then $p_0 \in X = \text{set}(T)$ if there exists a pair of constants $\beta \in (0, 1)$ and $C > 0$ such that \bar{X} has the tetrahedral property at p_0 for all sufficiently small $r > 0$.

Proof. By Theorem 3.38 we have

$$(241) \quad \liminf_{r \rightarrow 0} \frac{\|T\|(B_p(r))}{r^m} \geq C(2\beta)^{m-1} > 0$$

so $p \in \text{set}(T)$. \square

3.12. Fillings, Slices and Intervals. Section added in August 2012. Needs a second check.

In the above sections, a key step consisted of estimating $\mathbf{M}(M) \geq \text{FillVol}(\partial M)$. This is only a worthwhile estimate when $\partial M \neq 0$ or has a filling volume close to the mass.

A better estimate can be obtained using the following trick. Given a Riemannian manifold M ,

$$(242) \quad \text{Vol}(M) = \text{Vol}(M \times I) \geq \text{FillVol}(\partial(M \times I))$$

where the metric on $M \times I$ is defined in (140). This has the advantage that $M \times I$ is always a manifold with boundary. It may also be worthwhile to use an interval, I_ϵ , of length ϵ , then

$$(243) \quad \text{Vol}(M) = \frac{\text{Vol}(M \times I_\epsilon)}{\epsilon} \geq \frac{\text{FillVol}(\partial(M \times I_\epsilon))}{\epsilon}.$$

Intuitively it would seem that taking $\epsilon \rightarrow 0$ we converge to an equality.

We introduce the following notion made rigorous on arbitrary integral current spaces $M = (X, d_X, T)$ by applying Definition 3.5 and Proposition ??.

Definition 3.43. Given any $\epsilon > 0$, we define the ϵ interval filling volume,

$$(244) \quad \text{IFV}_\epsilon(M) = \text{FillVol}(\partial(M \times I_\epsilon)).$$

Lemma 3.44. Given an integral current space $M = (X, d, T)$,

$$(245) \quad \mathbf{M}(M) = \epsilon^{-1} \mathbf{M}(M \times I_\epsilon) \geq \epsilon^{-1} \text{IFV}_\epsilon(M).$$

Proof. This follows immediately from Proposition 3.7. \square

Definition 3.45. Given an integral current space, $M^m = (X, d, T)$ and $F_1, F_2, \dots, F_k : M \rightarrow \mathbb{R}$ with $k \leq m$ are Lipschitz functions with Lipschitz constant $\text{Lip}(F_j) = \lambda_j$ then for all $\epsilon > 0$ we can define the ϵ sliced interval filling volume of $\partial S(p, r) \in \mathbf{I}_{m-1}(\bar{X})$ to be

$$(246) \quad \mathbb{SIF}_\epsilon(p, r, F_1, \dots, F_k) = \prod_{j=1}^k \epsilon^{-1} \int_{t \in A_r} \text{FillVol}(\partial(\text{Slice}(S(p, r), F, t) \times I_\epsilon)) \mathcal{L}^k$$

where

$$(247) \quad A_r = [\min F_1, \max F_1] \times [\min F_2, \max F_2] \times \dots \times [\min F_k, \max F_k]$$

where $\min F_j = \min\{F(x) : x \in \bar{B}_p(r)\}$ and $\max F_j = \max\{F(x) : x \in \bar{B}_p(r)\}$. When the F_i are distance functions ρ_{p_i} we write,

$$(248) \quad \mathbb{SIF}_\epsilon(p, r, p_1, \dots, p_k) := \mathbb{SIF}_\epsilon(p, r, \rho_{p_1}, \dots, \rho_{p_k}).$$

Proposition 3.46. Given an integral current space, $M^m = (X, d, T)$ and $F_1, F_2, \dots, F_k : M \rightarrow \mathbb{R}$ with $k \leq m$ are Lipschitz functions with Lipschitz constant $\text{Lip}(F_j) = \lambda_j$ then for all $\epsilon > 0$ we can bound the mass of a ball in M as follows:

$$(249) \quad \mathbf{M}(S(p, r)) \geq \prod_{j=1}^k \lambda_j^{-1} \epsilon^{-1} \mathbb{SIF}_\epsilon(p, r, F_1, \dots, F_k).$$

Thus for any $p_1, \dots, p_k \in X$, we have

$$(250) \quad \mathbf{M}(S(p, r)) \geq \epsilon^{-1} \mathbb{SIF}_\epsilon(p, r, p_1, \dots, p_k).$$

Proof. By Proposition 3.10, Lemma 3.44 and Remark ?? we have

$$(251) \quad \mathbf{M}(S(p, r)) \geq \prod_{j=1}^k \lambda_j^{-1} \mathbf{M}(S(p, r) \llcorner dF_j)$$

$$(252) \quad \geq \prod_{j=1}^k \lambda_j^{-1} \int_{t \in \mathbb{R}^k} \mathbf{M}(\text{Slice}(S(p, r), F, t)) \mathcal{L}^k$$

$$(253) \quad = \prod_{j=1}^k \lambda_j^{-1} \int_{t \in A} \mathbf{M}(\text{Slice}(S(p, r), F, t)) \mathcal{L}^k$$

$$(254) \quad \geq \prod_{j=1}^k \lambda_j^{-1} \epsilon^{-1} \int_{t \in A} \text{FillVol}(\text{Slice}(S(p, r), F, t) \times I_\epsilon) \mathcal{L}^k.$$

□

Corollary 3.47. Thus $p \in \bar{X}$ lies in $X = \text{set}(T)$ iff there exists $\epsilon > 0$ such that

$$(255) \quad \liminf_{r \rightarrow 0} \frac{1}{\epsilon r^m} \int_{t \in A_r} \text{FillVol}(\partial(\text{Slice}(S(p, r), F, t) \times I_\epsilon)) \mathcal{L}^k > 0.$$

Corollary 3.48. Also $p \in \bar{X}$ lies in $X = \text{set}(T)$ iff there exists $C > 0$ such that

$$(256) \quad \liminf_{r \rightarrow 0} \frac{1}{C r^{m+1}} \int_{t \in A_r} \text{FillVol}(\partial(\text{Slice}(S(p, r), F, t) \times I_{Cr})) \mathcal{L}^k > 0.$$

Corollary 3.49.

$$(257) \quad \mathbf{M}(M) \geq \prod_{j=1}^k \lambda_j^{-1} \epsilon^{-1} \int_{t \in \mathbb{R}^k} \text{FillVol}(\partial(\text{Slice}(M, F, t) \times I_\epsilon)) \mathcal{L}^k$$

3.13. **Geodesics.** I have two subsections on geodesics that need to be texed and inserted here.

4. CONVERGENCE

In this section we examine the properties of sequences of integral current spaces and Riemannian manifolds which converge in the intrinsic flat sense to an integral current space. Recall Theorem 2.40.

Definition 4.1. If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then we say $x_i \in X_i$ are a converging sequence that converge to $x_\infty \in X_\infty$ if there exists a complete metric space Z and isometric embeddings $\varphi_i : M_i \rightarrow Z$ such that $\varphi_{i\#}T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#}T_\infty$ and $\varphi_i(x_i) \rightarrow \varphi_\infty(x_\infty)$. If we say collection of points, $\{p_{1,i}, p_{2,i}, \dots, p_{k,i}\}$, converges to a corresponding collection of points, $\{p_{1,\infty}, p_{2,\infty}, \dots, p_{k,\infty}\}$, if $\varphi_i(p_{j,i}) \rightarrow \varphi_\infty(p_{j,\infty})$ for $j = 1..k$.

Here we prove every point $x \in X_\infty$ has a sequence $x_i \in X_i$ converging to x in this sense [Theorem 4.3]. We then discuss uniqueness of limit points and Bolzano-Weierstrass Theorems which are similar to corresponding theorems in the setting of Gromov-Hausdorff convergence. Unlike in Gromov-Hausdorff convergence we have the possibility of disappearing sequences of points:

Definition 4.2. If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then we say $x_i \in X_i$ are Cauchy if there exists a complete metric space Z and isometric embeddings $\varphi_i : M_i \rightarrow Z$ such that $\varphi_{i\#}T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#}T_\infty$ and $\varphi_i(x_i) \rightarrow z_\infty \in Z$. We say the sequence is disappearing if $z_\infty \notin \varphi_\infty(X_\infty)$.

Examples with disappearing splines from [20] demonstrate that there exist Cauchy sequences with no limit.

If $(X_i, d_i) \xrightarrow{\text{GH}} (Y, d_Y)$, then one can see that every point $y \in Y$ has a sequence $x_i \in X_i$ converging to it which is Cauchy in the sense of Definition 4.2. Then $Y = X_\infty$ if every one of those Cauchy sequences converges. To prove the Cauchy sequence converges we must show

$$(258) \quad z_\infty \in \text{set}(\varphi_{\infty\#}(T_\infty))$$

which happens iff

$$(259) \quad \liminf_{r \rightarrow 0} \mathbf{M}(\varphi_{\infty\#}(T_\infty) \llcorner B(z_\infty, r)) / r^m > 0.$$

In the prior subsections we gave many means for estimating these masses using filling volumes and sliced filling volumes.

In [19], we stated without proof that balls converge. We provide the details of that proof here using the notion of converging and Cauchy sequences of points [Theorem 4.18]. We prove that slices of the spaces converge [Proposition 4.19]. We prove that the filling volumes and sliced filling volumes converge [Theorem 4.20]. These results are applied to prove when Cauchy sequences of points $x_i \in X_i$ converge to a point $x \in X$.

We prove the intrinsic flat and Gromov-Hausdorff limits of spaces with uniform tetrahedral properties or uniform lower bounds, $\mathbf{SF}_k(p, r) \geq C_{SF} r^m$, agree [Theorems 4.31 and 4.30].

We summarize consequences of continuity of these properties on certain classes of spaces and consequences of conjectures regarding intrinsic flat convergence.

Mention Arz Asco theorems and also various continuity theorems.

4.1. Limit Points and Diameter Lower Semicontinuity. These easy theorems were added in August 2012 in response to questions.

Recall Definition 4.1.

Theorem 4.3. *If a sequence of integral current spaces, $M_i = (X_i, d_i, T_i) \in \mathcal{M}_0^n$, converges to a integral current space, $M = (X, d, T) \in \mathcal{M}_0^n$, in the intrinsic flat sense, then every point z in the limit space M is the limit of points $x_i \in M_i$. In fact there exists a sequence of maps $F_i : X \rightarrow X_i$ such that $x_i = F_i(x)$ converges to x and*

$$(260) \quad \lim_{i \rightarrow \infty} d_i(F_i(x), F_i(y)) = d(x, y).$$

This sequence of maps F_i are not uniquely defined and are not even unique up to isometry.

Proof. By Theorem 2.40 there exists a common metric space Z and isometric embeddings $\varphi_i : X_i \rightarrow Z$ and $\varphi : X \rightarrow Z$ such that

$$(261) \quad \varphi_{\#} T - \varphi_{i\#} T_i = U_i + \partial V_i$$

where $m_i = \mathbf{M}(U_i) + \mathbf{M}(V_i) \rightarrow 0$. So $\varphi_{i\#} T_i$ converges in the flat and the weak sense to $\varphi_{\#} T$.

Let ρ_x be the distance function from $\varphi(x)$. Applying the Ambrosio-Kirchheim Slicing Theorem we know

$$(262) \quad S_{i,\epsilon,x} := \varphi_{i\#} T_i \llcorner \rho_x^{-1}([0, \epsilon)) \in \mathbf{I}_m(Z)$$

$$(263) \quad T_{\epsilon,x} := \varphi_{\#} T \llcorner \rho_x^{-1}([0, \epsilon)) \in \mathbf{I}_m(Z)$$

for almost every $\epsilon > 0$. Restricting ourselves to such an $\epsilon \in (0, 1/j)$, we have $S_{i,\epsilon,x}$ converges weakly to $T_{\epsilon,x}$. In particular, there exists $N_{\epsilon,x} \in \mathbb{N}$ large enough that $S_{i,\epsilon,x}$ is not zero for all $i \geq N_{\epsilon,x}$. So for all $x \in X$ and any $j \in \mathbb{N}$

$$(264) \quad \exists N_{j,x} \text{ s.t. } \exists s_{i,j,x} \in \text{set}(S_i) \cap B(x, 1/j) \quad \forall i \geq N_{j,x}.$$

Without loss of generality we assume $N_{j,x}$ is increasing in j . For $i \in \{1, \dots, N_{1,x}\}$ take $j_i = 1$. Then for $i \in \{N_{j-1,x} + 1, \dots, N_{j,x}\}$ let $j_i = j$. Thus $i \geq N_{j_i,x}$. Let

$$(265) \quad x_i = \varphi_i^{-1}(s_{i,j_i}).$$

Then $\varphi_i(x_i) \in B(x, 1/j_i)$ and $\varphi_i(x_i) \rightarrow \varphi(x)$.

Since this process can be completed for any $x \in X$, we have defined maps $F_i : X \rightarrow X_i$ such that

$$(266) \quad \varphi_i(F_i(x)) \rightarrow \varphi(x).$$

Finally

$$(267) \quad d_i(F_i(x), F_i(y)) = d_Z(\varphi_i(F_i(x)), \varphi_i(F_i(y))) \rightarrow d_Z(\varphi(x), \varphi(y)) = d(x, y).$$

□

Definition 4.4. *Like any metric space, one can define the diameter of an integral current space, $M = (X, d, T)$, to be*

$$(268) \quad \text{Diam}(M) = \sup \{d_X(x, y) : x, y \in X\} \in [0, \infty] .$$

However, we explicitly define the diameter of the 0 integral current space to be 0. A space is bounded if the diameter is finite.

Theorem 4.5. Suppose $M_i \xrightarrow{\mathcal{F}} M$ are integral current spaces then

$$(269) \quad \text{Diam}(M) \leq \liminf_{i \rightarrow \infty} \text{Diam}(M_i) \subset [0, \infty]$$

Proof. Note that by the definition, $\text{Diam}(M_i) \geq 0$, so the \liminf is always ≥ 0 . Thus the inequality is trivial when M is the 0 space. Assuming M is not the 0 space, we have for any $\epsilon > 0$, there exists $x, y \in X$ such that

$$(270) \quad \text{Diam}(M) \leq d(x, y) + \epsilon.$$

By Theorem 4.3, we have $x_i, y_i \in X_i$ converging to $x, y \in X$ so that

$$(271) \quad \text{Diam}(M) \leq \lim_{i \rightarrow \infty} d_i(x_i, y_i) + \epsilon \leq \liminf_{i \rightarrow \infty} \text{Diam}(X_i) + \epsilon.$$

□

4.2. Flat convergence to Gromov-Hausdorff Convergence. In this subsection, we prove Theorem 4.6:

Theorem 4.6. If a sequence of precompact integral current spaces, $M_i = (X_i, d_i, T_i) \in \mathcal{M}_0^m$, converges to a precompact integral current space, $M = (X, d, T) \in \mathcal{M}_0^m$, in the intrinsic flat sense, then there exists $S_i \in \mathbf{I}_m(\bar{X}_i)$ such that $N_i = (\text{set}(S_i), d_i)$ converges to (\bar{X}, d) in the Gromov-Hausdorff sense and

$$(272) \quad \liminf_{i \rightarrow \infty} \mathbf{M}(S_i) \geq \mathbf{M}(M).$$

When the M_i are Riemannian manifolds, the N_i can be taken to be settled completions of open submanifolds of M_i .

Remark 4.7. If in addition it is assumed that $\lim_{i \rightarrow \infty} \mathbf{M}(M_i) = \mathbf{M}(M)$, then by (272) we have $\mathbf{M}(\text{set}(T_i - S_i), d_i, T_i - S_i) = 0$. In the Riemannian setting, we have $\text{Vol}(M_i \setminus N_i) \rightarrow 0$.

Remark 4.8. In Ilmanen's example [20] of a sphere with increasingly many splines, the S_i may be chosen to be integration over the spherical part of M_i with balls around the tips removed. Then $\text{set}(S_i)$ are manifolds with boundary converging to the sphere in the Gromov-Hausdorff and intrinsic flat sense.

Remark 4.9. The precompactness of the limit integral current spaces is necessary in this theorem because a noncompact limit space can never be the Gromov-Hausdorff limit of precompact spaces, yet there are sequences of compact Riemannian manifolds, M_j , whose intrinsic flat limit is an unbounded complete Riemannian manifold of finite volume [20][Ex A.10] and another example of such spaces whose Intrinsic Flat limit is a bounded noncompact integral current space [20][Ex A.11].

Remark 4.10. Gromov's Compactness Theorem combined with Theorem 4.6 implies that that any sequence of $x_i \in N_i \subset M_i$ has a subsequence converging to a point x in the metric completion of M . Other points need not have limit points, as can be seen when the tips of thin splines disappear in the examples from [20].

We now prove Theorem 4.6:

Proof. By Theorem 2.40 there exists a common metric space Z and isometric embeddings $\varphi_i : X_i \rightarrow Z$ and $\varphi : X \rightarrow Z$ such that

$$(273) \quad \varphi_{\#} T - \varphi_{\#} T_i = U_i + \partial V_i$$

where $m_i = \mathbf{M}(U_i) + \mathbf{M}(V_i) \rightarrow 0$. So $\varphi_{\#} T_i$ converges in the flat and the weak sense to $\varphi_{\#} T$.

Since $M \in \mathcal{M}_0^m$, $\varphi(X)$ is precompact. Let $\rho : Z \rightarrow \mathbb{R}$ be the distance function from $\varphi(X)$.

By the Ambrosio-Kirchheim Slicing Theorem [Theorem 2.23], we know that

$$(274) \quad S_{i,\epsilon} := \varphi_{i\#} T_i \llcorner \rho^{-1}([0, \epsilon)) \in \mathbf{I}_m(Z)$$

for almost every $\epsilon > 0$. Fix any such ϵ .

Note that $\varphi_{\#} T = \varphi_{\#} T \llcorner \rho^{-1}[0, \epsilon)$. So both $\varphi_{i\#}(T_i)$ and $S_{i,\epsilon}$ converges weakly to $\varphi_{\#} T$ for fixed $\epsilon > 0$. In particular, by lower semicontinuity of mass we have

$$(275) \quad \forall \epsilon > 0 \exists M_{\epsilon} \in \mathbb{N} \text{ such that } \mathbf{M}(S_{i,\epsilon}) \geq \mathbf{M}(T) - \epsilon \quad \forall i \geq M_{\epsilon}.$$

By definition

$$(276) \quad \text{set}(S_{i,\epsilon}) \subset \bar{T}_{\epsilon}(\varphi(X)) \subset T_{2\epsilon}(\varphi(X)).$$

To prove the Hausdorff distance between $S_{i,\epsilon}$ and X is small we will prove

$$(277) \quad \varphi(X) \subset T_{2\epsilon}(\text{set}(S_{i,\epsilon})) \quad \forall i \geq N_{\epsilon}.$$

To prove (277), we first note that for any $x \in X$, we can let ρ_x be the distance function from $\varphi(x)$, and applying the slicing theorem again we know

$$(278) \quad S_{i,\epsilon,x} := \varphi_{i\#} T_i \llcorner \rho_x^{-1}([0, \epsilon)) \in \mathbf{I}_m(Z)$$

$$(279) \quad T_{\epsilon,x} := \varphi_{i\#} T \llcorner \rho_x^{-1}([0, \epsilon)) \in \mathbf{I}_m(Z)$$

for almost every $\epsilon > 0$. Restricting ourselves to such an $\epsilon > 0$, we have $S_{i,\epsilon,x}$ converges weakly to $T_{\epsilon,x}$. In particular, there exists $N_{\epsilon,x} \in \mathbb{N}$ large enough that $S_{i,\epsilon,x}$ is not zero for all $i \geq N_{\epsilon,x}$. So for all $x \in X$ and almost every $\epsilon > 0$

$$(280) \quad \exists N_{\epsilon,x} \text{ s.t. } \exists S_{i,\epsilon,x} \in \text{set}(S_i) \cap B(x, \epsilon) \quad \forall i \geq N_{\epsilon,x}.$$

Note that for all $\epsilon > 0$ and all $x \in X$ we have (280).

By the precompactness of X , there is a finite ϵ net, $X_{\epsilon} = \{x_1, \dots, x_N\}$ on $\varphi(X)$ (i.e. the union of $B(x_i, \epsilon)$ contains X_{ϵ}) and choosing $N_{\epsilon} = \max\{N_{\epsilon,x_j} : x_j \in X_{\epsilon}\}$, we know every $x \in X$, there exists $x_j \in X_{\epsilon}$ such that for all $i \geq N_{\epsilon}$ there exists

$$(281) \quad s_x := s_{i,\epsilon,x_j} \in \text{set}(S_{i,\epsilon}) \text{ s.t. } d_Z(s_x, \varphi(x)) < 2\epsilon$$

so we have proven (277). Combining this with (276) we have the Hausdorff distance

$$(282) \quad d_H^Z(\text{set}(S_{i,\epsilon}), \varphi(X)) \leq 2\epsilon \quad \forall i \geq N_{\epsilon}.$$

We may choose $N_{\epsilon} \geq M_{\epsilon}$ of (275).

Let $\epsilon_k \rightarrow 0$ be a decreasing sequence of ϵ for which all these currents are defined. Let $N_k := N_{\epsilon_k}$. Let

$$(283) \quad S_i = T_i \in \mathbf{I}_m(X_i) \text{ for } i = 1 \text{ to } N_1$$

$$(284) \quad S_i = \varphi_{i\#}^{-1} S_{i,\epsilon_1} \in \mathbf{I}_m(X_i) \text{ for } i = N_1 + 1 \text{ to } N_2$$

and so on:

$$(285) \quad S_i = \varphi_{i\#}^{-1} S_{i,\epsilon_j} \in \mathbf{I}_m(X_i) \text{ for } i = N_j + 1 \text{ to } N_{j+1}$$

Then by (282) we have Gromov-Hausdorff convergence:

$$(286) \quad d_{GH}^Z(\text{set}(S_i), \varphi(X)) \leq 2\epsilon_i.$$

By (275) we also have

$$(287) \quad \mathbf{M}(S_i) \geq \mathbf{M}(T) - \epsilon_i$$

which gives us (273). \square

Remark 4.11. *One could construct a common metric space Z for Examples A.10 and A.11 of [20] and find $S_{i,\epsilon}$ as in the above proof satisfying (276). However, in that example, (277) will fail to hold. This is where the precompactness of the limit space is essential in the proof.*

Remark 4.12. *Examples in [20] demonstrate that the metric space of a current space need not be a length space. In general, when a sequence of Riemannian manifolds converges in the flat sense to a current space it need not be a geodesic length space. Here we see that if the set (S_i) are length spaces or approximately length spaces, then the limit current space is in fact a length space. This occurs for example in Ilmanen's example of [20]. It also occurs whenever the Gromov-Hausdorff limits and flat limits of length spaces agree which we will examine further below. It might be interesting to develop a notion of an approximate length space that suffices to give a geodesic limit space. What properties must hold on M_i to guarantee that their limit is a geodesic length space?*

Remark 4.13. *It is not immediately clear whether the integral current spaces, N_i , constructed in the proof of Theorem 4.6 actually converge in the intrinsic flat sense to M . One expects an extra assumption on total mass would be needed to interchange between flat and weak convergence, but even so it is not completely clear. One would need to uniformly control the masses of ∂N_i using a common upper bound on $\mathbf{M}(N)$ which can be done using theorems in Section 5 of [1], but is highly technical. It is worth investigating.*

4.3. Limits of Slices and Balls. Here we prove that well chosen slices and balls about converging points converge.

Proposition 4.14. *Given an integral current space, $M = (X, d, T)$ and Lipschitz functions, $\rho : X \rightarrow \mathbb{R}$ and $f : X \rightarrow \mathbb{R}$ such that*

$$(288) \quad |f(x) - \rho_p(x)| < \delta \quad \forall x \in X,$$

then for almost every $r \in \mathbb{R}$

$$(289) \quad d_{\mathcal{F}}(\text{Slice}(M, \rho, r), \text{Slice}(M, f, r)) \leq \|T\|(\rho^{-1}(r - \delta, r + \delta)) + \|\partial T\|(\rho^{-1}(r - \delta, r + \delta)).$$

Proof. First observe that by the definition of intrinsic flat distance,

$$d_{\mathcal{F}}(\text{Sphere}(p, r), \text{Slice}(M, f, r)) \leq d_F^{\bar{X}}(< T, \rho, r >, < T, f, r >) \leq \mathbf{M}(B) + \mathbf{M}(A)$$

where

$$(290) \quad B = T \llcorner \rho^{-1}(-\infty, r] - T \llcorner f^{-1}(-\infty, r]$$

$$(291) \quad A = (\partial T) \llcorner f^{-1}(-\infty, r] - (\partial T) \llcorner \rho^{-1}(-\infty, r].$$

Next note that for any pair of sets $U, V \subset X$,

$$(292) \quad \mathbf{M}(T \llcorner U - T \llcorner V) = \mathbf{M}(T \llcorner (\chi_U - \chi_V)) = \mathbf{M}(T \llcorner (U \setminus V)) + \mathbf{M}(T \llcorner (V \setminus U))$$

and the same holds for ∂T . Since

$$(293) \quad \rho^{-1}(-\infty, r] \setminus f^{-1}(-\infty, r] \subset \rho^{-1}(r - \delta, r + \delta)$$

and

$$(294) \quad f^{-1}(-\infty, r] \setminus \rho^{-1}(-\infty, r] \subset \rho^{-1}(r - \delta, r + \delta)$$

we have

$$(295) \quad \mathbf{M}(B) \leq \mathbf{M}(T \llcorner (\rho^{-1}(r - \delta, r + \delta)))$$

$$(296) \quad \mathbf{M}(A) \leq \mathbf{M}(\partial T \llcorner (\rho^{-1}(r - \delta, r + \delta))).$$

□

Proposition 4.15. *Given two integral current spaces, $M_i = (X_i, d_i, T_i)$, and isometric embeddings $\varphi_i : X_i \rightarrow Z$, and points $p_i \in X_i$, then*

$$(297) \quad d_{\mathcal{F}}(S(p_1, r), S(p_2, r)) \leq d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2)$$

$$(298) \quad + \|T_2\|(\rho_{p_2}^{-1}(r - \delta, r + \delta)) + \|\partial T_2\|(\rho_{p_2}^{-1}(r - \delta, r + \delta))$$

where $\delta = d_Z(\varphi_1(p_1), \varphi_2(p_2))$. More generally, if we have points $z_i \in Z$ then

$$(299) \quad d_F^Z(\varphi_{1\#} \text{Slice}(M_1, \rho_{z_1} \circ \varphi_1, r), \varphi_{2\#}(\text{Slice}(M_2, \rho_{z_2} \circ \varphi_2, r)) \leq$$

$$(300) \quad \leq d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2) + \|T_2\|(\varphi_2^{-1}\rho_{z_2}^{-1}(r - \delta, r + \delta)) + \|\partial T_2\|(\varphi_2^{-1}\rho_{z_2}^{-1}(r - \delta, r + \delta)).$$

where $\delta = d_Z(z_1, z_2)$.

Proof. By (66), if $F : Z \rightarrow \mathbb{R}^k$ is Lipschitz, then for almost every $t \in \mathbb{R}^k$,

$$(301) \quad \langle \varphi_{\#}T, F, t \rangle = \varphi_{\#} \langle T, F \circ \varphi, t \rangle.$$

By (57), we know that, if

$$(302) \quad \langle \varphi_{1\#}T_1, F, t \rangle - \langle \varphi_{2\#}T_2, F, t \rangle = \langle \partial B, F, t \rangle + \langle A, F, t \rangle$$

$$(303) \quad = (-1)^k \partial \langle B, F, t \rangle + \langle A, F, t \rangle$$

Combining these two facts, we have

$$\begin{aligned} d_F^Z(\varphi_{1\#} \langle T_1, F \circ \varphi_1, t \rangle, \varphi_{2\#} \langle T_2, F \circ \varphi_2, t \rangle) &\leq \mathbf{M}(\langle B, F, t \rangle) + \mathbf{M}(\langle A, F, t \rangle) \\ &\leq \prod_{j=1}^k \text{Lip}(F_j)(\mathbf{M}(B) + \mathbf{M}(A)) \\ &\leq \prod_{j=1}^k \text{Lip}(F_j)(d_{\mathcal{F}}(M_1, M_2)). \end{aligned}$$

Now we apply this to the function $F(z) = \rho_{z_1}(z)$. Then setting $f = \rho_{z_1} \circ \varphi_2$ we have

$$(304) \quad d_F^Z(\langle T_1, \rho_{z_1} \circ \varphi_1, t \rangle, \langle T_2, f, t \rangle) \leq d_{\mathcal{F}}(M_1, M_2).$$

By the triangle inequality on Z , the function $f : X_2 \rightarrow \mathbb{R}$ satisfies (288) for $M = M_2$ and $\rho = \rho_{z_2} \circ \varphi_2$. Thus we have (289). Combining this with (304) gives (299).

Finally (299) implies (297) by taking $z_i = \varphi_1(p_i)$ and noting that $\rho_{z_i} \circ \varphi_i = \rho_{p_i}$. □

Lemma 4.16. *Given an integral current space, (X, d, T) , and any Lipschitz function, $f : X \rightarrow \mathbb{R}$, we have*

$$(305) \quad \lim_{\delta \rightarrow 0} \|T\| \left(f^{-1}(r - \delta, r + \delta) \right) = 0$$

Proof. In fact, given any bounded Borel measure μ on a complete metric space Z such that $\mu(Z) < \infty$ and any Lipschitz function $f : Z \rightarrow \mathbb{R}$, we have for almost every $r \in \mathbb{R}$:

$$(306) \quad \lim_{\delta \rightarrow 0} \mu(f^{-1}(r - \delta, r + \delta)) = 0.$$

Assume not. Then there exists $A \subset \mathbb{R}$, such that $\mathcal{L}^1(A) > 0$ where (306) fails. Since A is the countable union of sets A_k where the limit in (306) is greater than $1/k$, we know there

exists k such that $\mathcal{L}^1(A_k)$. Let $N = 2k\mu(Z)$, then there exists $r_1 < r_2 < \dots < r_N$ all in A_k and $\delta_j > 0$ such that

$$(307) \quad \mu(f^{-1}(r - \delta, r + \delta)) > 1/k \quad \forall \delta < \delta_j.$$

Then taking $\delta'_j < \delta_j$ such that the intervals $(r_j - \delta'_j, r_j + \delta'_j)$ are disjoint, we have

$$(308) \quad \mu(Z) \geq \sum_{j=1}^N \mu(f^{-1}(r_j - \delta'_j, r_j + \delta'_j))$$

$$(309) \quad \geq \sum_{j=1}^N (1/k) = N(1/k) = 2\mu(Z)$$

which is a contradiction. \square

Proposition 4.17. *Suppose we have a sequence of integral current spaces, $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i : X_i \rightarrow Z$, such that*

$$(310) \quad \lim_{i \rightarrow \infty} d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0$$

and points $z_i \in Z$ such that $\delta_i = d_Z(z_i, z_\infty)$, then setting

$$(311) \quad S_i = \varphi_{i\#} \text{Slice}(M_i, \rho_{z_i} \circ \varphi_i, r)$$

then for almost every $r \in \mathbb{R}$ we have

$$(312) \quad d_F^Z(S_i, S_\infty) \leq d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty)$$

$$(313) \quad + \|T_\infty\| \left(f^{-1}(r - \delta_i, r + \delta_i) \right) + \|\partial T_\infty\| \left(f^{-1}(r - \delta_i, r + \delta_i) \right)$$

$$(314)$$

where $f(x) = \rho_{z_\infty}(\varphi_\infty(x))$. If $\delta_i \rightarrow 0$, then for almost every $r \in \mathbb{R}$,

$$(315) \quad \lim_{i \rightarrow \infty} d_F^Z(S_i, S_\infty) = 0.$$

In our applications this proposition will be applied to $z_i = \varphi_i(p_i)$ where $p_i \in X_i$ are converge (or are Cauchy) in which case $\rho_{z_i}(\varphi_i(x)) = \rho_{p_i}(x)$. Thus, spheres

$$(316) \quad d_{\mathcal{F}}(\text{Sphere}(p_i, r), \text{Sphere}(p_\infty, r)) \rightarrow 0 \text{ for almost every } r \in \mathbb{R}$$

if $p_i \in X_i$ converge to $p_\infty \in X_\infty$. This convergence of spheres was applied without an explicit proof in [19]. The precise bound obtained here is needed to study limits of sliced filling volumes.

Proof. We know that for almost every $r \in \mathbb{R}$, the slices are well defined. First we may apply Proposition 4.15 with $M_i = M_1$ and $M_\infty = M_2$. Note that

$$(317) \quad \rho_{z_\infty}(\varphi(x)) = \rho_{p_\infty}(x) \quad \forall x \in X_\infty.$$

The rest follows from Lemma 4.16. \square

Proposition 4.18. *Suppose we have a sequence of integral current spaces, $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i : X_i \rightarrow Z$, such that*

$$(318) \quad d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0$$

and points $z_i \in Z$ such that $\delta_i = d_Z(z_i, z_\infty)$, then for almost every $r \in \mathbb{R}$ the balls, $S_i = \varphi_{i\#}T_i \llcorner B(z_i, r)$, satisfy

$$(319) \quad d_F^Z(S_i, S_\infty) \leq 2d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty)$$

$$(320) \quad + \|T_\infty\| \left(f^{-1}(r - \delta_i, r + \delta_i) \right)$$

where $f(x) = \rho_{z_\infty}(\varphi_\infty(x))$. If $\delta_i \rightarrow 0$, then for almost every $r \in \mathbb{R}$,

$$(321) \quad \lim_{i \rightarrow \infty} d_F^Z(S_i, S_\infty) = 0.$$

In our applications this proposition will be applied to $z_i = \varphi_i(p_i)$ where $p_i \in X_i$ are converge (or are Cauchy) in which case $\rho_{z_i}(\varphi_i(x)) = \rho_{p_i}(x)$. Thus, balls

$$(322) \quad d_{\mathcal{F}}(S(p_i, r), S(p_\infty, r)) \rightarrow 0 \text{ for almost every } r \in \mathbb{R}$$

if $p_i \in X_i$ converge to $p_\infty \in X_\infty$. This convergence of balls was applied without an explicit proof in [19].

Proof. There exists integral currents A_i, B_i in Z , such that

$$(323) \quad \varphi_{\#}T_i - \varphi_{\infty\#}T_\infty = A_i + \partial B_i$$

and

$$(324) \quad d_F^Z(\varphi_{\#}T_i, \varphi_{\infty\#}T_\infty) = \mathbf{M}(A_i) + \mathbf{M}(B_i).$$

For almost every r , the restrictions of these spaces to balls, $B(z_i, r)$, are integral current spaces such that

$$(325) \quad (\varphi_{\#}T_i - \varphi_{\infty\#}T_\infty) \llcorner \bar{B}(z_i, r) = A_i \llcorner \bar{B}(z_i, r) + (\partial B_i) \llcorner \bar{B}(z_i, r)$$

$$(326) \quad = A_i \llcorner \bar{B}(z_i, r) + \langle B_i, \rho_i, r \rangle + \partial(B_i \llcorner \bar{B}(z_i, r))$$

Since

$$(327) \quad \mathbf{M}(A_i \llcorner \bar{B}(z_i, r) + \langle B_i, \rho_i, r \rangle) \leq \mathbf{M}(A_i) + \mathbf{M}(B_i),$$

$$(328) \quad \mathbf{M}(B_i \llcorner \bar{B}(z_i, r)) \leq \mathbf{M}(B_i)$$

and

$$(329) \quad \mathbf{M}(\varphi_{\infty\#}T_\infty \llcorner \bar{B}(z_i, r) - \varphi_{\infty\#}T_\infty \llcorner \bar{B}(z_\infty, r)) \leq \|\varphi_{\infty\#}T_\infty\| \text{Ann}_{z_\infty}(r - \delta_i, r + \delta_i)$$

we have our first claim. The rest follows from Lemma 4.16. \square

Proposition 4.19. Suppose we have a sequence of m dimensional integral current spaces, $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i : X_i \rightarrow Z$, such that

$$(330) \quad \lim_{i \rightarrow \infty} d_F^Z(\varphi_{\#}T_i, \varphi_{\infty\#}T_\infty) = 0$$

and points $z_{j,i} \in Z$ such that $\delta_{j,i} = d_Z(z_{j,i}, z_{j,\infty})$ for $j = 1..k \leq m$ then

$$(331) \quad d_F^Z(\varphi_{\#} \text{Slice}(M_i, \rho_i, r), \varphi_{\infty\#}(\text{Slice}(M_\infty, \rho_\infty, r)) \leq d_F^Z(\varphi_{\#}T_i, \varphi_{\infty\#}T_\infty) + \sum_{j=1}^k (a_j(r_1, \dots, r_k) + b_j(r_1, \dots, r_k))$$

where

$$(332) \quad a_j(r_1, \dots, r_k) = \|(\text{Slice}(T_\infty, f_1, \dots, f_j, r_1, \dots, r_j)) \llcorner \left(\bigcap_{n=1}^j (f_n^{-1}(r - \delta_{n,i}, r + \delta_{n,i})) \right)\|$$

and

$$(333) \quad b_j(r_1, \dots, r_k) = \|(\text{Slice}(\partial T_\infty, f_1, \dots, f_j, r_1, \dots, r_j)) \llcorner \left(\bigcap_{n=1}^j (f_i^{-1}(r - \delta_{n,i}, r + \delta_{n,i})) \right)\|$$

where $f_i(x) = \rho_{z_{\infty,i}}(\varphi_\infty(x))$. If $\lim_{i \rightarrow \infty} \delta_{j,i} = 0$ then for almost every $r \in \mathbb{R}^k$,

$$(334) \quad \lim_{i \rightarrow \infty} d_F^Z(\varphi_{\#} \text{Slice}(M_i, \rho_i, r), \varphi_{\infty\#}(\text{Slice}(M_\infty, \rho_\infty, r)) = 0$$

where $\rho_i : M_i \rightarrow \mathbb{R}^k$ is defined by

$$(335) \quad \rho_i(x) = (\rho_{z_{1,i}}(\varphi_i(x)), \rho_{z_{2,i}}(\varphi_i(x)), \dots, \rho_{z_{k,i}}(\varphi_i(x))).$$

In our applications this proposition will be applied to $z_{j,i} = \varphi_i(p_{j,i})$ where $p_{j,i} \in X_i$ in which case $\rho_{z_{j,i}}(\varphi_i(x)) = \rho_{p_{j,i}}(x)$.

Proof. We will prove this by induction on k . The base step has been proven in Proposition 4.17. By the induction hypothesis, for almost every $r \in \mathbb{R}$, we have a sequence of the integral current spaces $N_i = \text{Slice}(M_i, \rho_i, r)$ such that

$$(336) \quad d_F^Z(\varphi_{1\#}N_i, \varphi_{\infty\#}N_\infty) = d_{F,i,k}$$

We introduce two new points $z_{k+1,i} \in Z$ and set

$$(337) \quad \delta_{k+1,i} = d_Z(z_{k+1,i}, z_{k+1,\infty}).$$

Applying Proposition 4.17, we see that for almost every $r_{k+1} \in \mathbb{R}$ we have

$$(338) \quad d_{F,i,(k+1)} \leq d_{F,i,k} + a_{k+1} + b_{k+1}$$

The first inequality in the proposition follows from

$$(339) \quad \text{Slice}(N, \rho_{k+1}, r_{k+1}) = \text{Slice}(\text{Slice}(M, \rho_1, \dots, \rho_k, r_1, \dots, r_k), \rho_{k+1}, r_{k+1})$$

$$(340) \quad = \text{Slice}(M, \rho_1, \dots, \rho_{k+1}, r_1, \dots, r_{k+1}).$$

For almost every $r \in \mathbb{R}^k$, the k^{th} slice of T_∞ is an integral current space, so we may apply Lemma 4.16, to obtain the limit. \square

4.4. Continuity of Sliced Filling Volumes. Recall that Theorem 2.46 implies the continuity of filling volume in the following sense:

$$(341) \quad M_i \xrightarrow{\mathcal{F}} M_\infty \implies \text{FillVol}(\partial M) \rightarrow \text{FillVol}(\partial M)$$

In this section we combine Theorem 2.46 in combination with the convergence of slices proven in Proposition 4.17 and Proposition 4.19. An immediate consequence of these results is that the filling volumes of slices converge. In particular the filling volumes of spheres converge to the filling volumes spheres, as stated in [20]. Finally we prove the sliced filling volumes are continuous via an application of the Dominated Convergence Theorem of Real Analysis.

Theorem 4.20. *Suppose $M_i \xrightarrow{\mathcal{F}} M$ and $\text{Diam}(M_i) \leq D_0$. If $p_i \in M_i$ converge to $p_\infty \in M_\infty$, and $q_{j,i} \in M_\infty$ converge to $q_{j,\infty} \in M_\infty$ for $j = 1..k$ then*

$$(342) \quad \lim_{i \rightarrow \infty} \mathbf{SF}(p_i, r, q_{1,i}, \dots, q_{k,i}) = \mathbf{SF}(p_\infty, r, q_{1,\infty}, \dots, q_{k,\infty}).$$

and thus

$$(343) \quad \liminf_{i \rightarrow \infty} \mathbf{SF}_k(p_i, r) \geq \mathbf{SF}_k(p_\infty, r).$$

Before proving this theorem we first state and prove the following technical proposition:

Proposition 4.21. *Suppose we have a sequence of m dimensional integral current spaces, $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i : X_i \rightarrow Z$, such that*

$$(344) \quad d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0$$

and points $z_{j,i} \in Z$ such that $\delta_{j,i} = d_Z(z_{j,i}, z_{j,\infty})$ for $j = 1..k$ for some $k \in \{0, \dots, m-1\}$ and $p_i \in X_i$, such that $\delta_i = d_Z(\varphi_i(p_i), \varphi(p))$, then for almost every $t \in \mathbb{R}^k$

$$(345) \quad |\text{FillVol}(\partial \text{Slice}(M_i, \rho_i, t)) - \text{FillVol}(\partial \text{Slice}(M_\infty, \rho_\infty, t))| \leq d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) + \sum_{j=1}^k (a_j(t) + b_j(t))$$

where a_j, b_j are defined in (332)-(333) and ρ_i is defined in (335). If $\lim_{i \rightarrow \infty} \delta_{j,i} = 0$ then for almost every $r \in \mathbb{R}$ and $r \in \mathbb{R}^k$, the filling volumes of slices converge,

$$(346) \quad \lim_{i \rightarrow \infty} \text{FillVol}(\partial \text{Slice}(M_i, \rho_i, t)) = \text{FillVol}(\partial \text{Slice}(M_\infty, \rho_\infty, t))$$

while the masses satisfy

$$(347) \quad \liminf_{i \rightarrow \infty} \mathbf{M}(\text{Slice}(M_i, \rho_i, t)) \geq \mathbf{M}(\text{Slice}(M_\infty, \rho_\infty, t)).$$

Furthermore, if $\text{Diam}(M_i) \leq D_0$ then

$$(348) \quad \lim_{i \rightarrow \infty} \int_{t \in [-R, R]^k} \text{FillVol}(\partial \text{Slice}(M_i, \rho_i, t)) \mathcal{L}^k = \int_{t \in [-R, R]^k} \text{FillVol}(\partial \text{Slice}(M_\infty, \rho_\infty, t)) \mathcal{L}^k.$$

where

$$(349) \quad R = 2D_0 + \sup\{\delta_{j,i} : i = 1..\infty, j = 1..k\} < \infty$$

Proof. Our first claim and the limits of the filling volumes of slices and the lower semicontinuity of the mass follows immediately from Proposition 4.19 combined with Theorem 2.46 and Theorem 2.42 respectively. In fact, we have (345). Since

$$\begin{aligned} \int_{[-R, R]^k} \text{FillVol}(\partial \text{Slice}(M_\infty, \rho_i, t)) \mathcal{L}^k &\leq \int_{[-R, R]^k} \mathbf{M}(\text{Slice}(M_\infty, \rho_i, t)) \mathcal{L}^k \\ &\leq \mathbf{M}(M_\infty) \\ \int_{[-R, R]^k} a_j(t) \mathcal{L}^k &\leq \int_{[-R, R]^k} \mathbf{M}(\text{Slice}(T_\infty, \rho_\infty, t)) \mathcal{L}^k \\ &\leq \mathbf{M}(M_\infty) \\ \int_{[-R, R]^k} b_j(t) \mathcal{L}^k &\leq \int_{[-R, R]^k} \mathbf{M}(\text{Slice}(\partial T_\infty, \rho_\infty, t)) \mathcal{L}^k \\ &\leq \mathbf{M}(\partial M_\infty). \end{aligned}$$

Thus we may apply the Dominated Convergence Theorem of Real Analysis to obtain the limit of the integrals. \square

We may now prove Theorem 4.20:

Proof. The convergence of the Sliced Filling Volumes with respect to converging points follows from Proposition 4.18 and Definition 3.20 and Theorem 4.21 stated and proven below. The lower semicontinuity of the k^{th} sliced filling then follows taking $q_{j,\infty}$ close to achieving the supremum in the definition of $\mathbf{SF}_k(p_\infty, r)$ and $q_{j,i} \rightarrow q_{j,\infty}$ so that so that

$$(350) \quad \liminf_{i \rightarrow \infty} \mathbf{SF}_k(p_i, r) \geq \lim_{i \rightarrow \infty} \mathbf{SF}_k(p_i, r, q_{j,i}) = \mathbf{SF}_k(p_\infty, r, q_{j,\infty}) = \mathbf{SF}_k(p_\infty, r) - \epsilon$$

and then taking $\epsilon \rightarrow 0$. \square

Remark 4.22. When Z is compact, take $q_{j,i}$ close to achieving the supremum in the definition of $\mathbf{SF}_k(p_i, r)$ so that

$$(351) \quad \lim_{i \rightarrow \infty} |\mathbf{SF}_k(p_i, r) - \mathbf{SF}_k(p_i, r, q_{j,i})| = 0.$$

By the compactness, a subsequence of the $q_{j,i} \rightarrow q_{j,\infty}$, so

$$(352) \quad \lim_{i \rightarrow \infty} \mathbf{SF}_k(p_i, r) = \lim_{i \rightarrow \infty} \mathbf{SF}(p_i, r, q_{j,i}) = \mathbf{SF}(p_\infty, r, q_{j,\infty}) \leq \mathbf{SF}_k(p_\infty, r).$$

4.5. Continuity of Interval Filling Volumes. Section added in August 2012 (needs a second check):

Recall the definition of the interval filling volume of a manifold or integral current space in Definition 3.43,

$$(353) \quad \mathbb{IFV}_{\epsilon\epsilon}(M) = \text{FillVol}(\partial(M \times I_\epsilon)) \leq \epsilon \mathbf{M}(M).$$

This notion was particularly useful for M without boundary. In this section we prove the interval filling volume is continuous with respect intrinsic flat convergence [Theorem 4.23]. Taking more precise estimates we prove the sliced interval filling volumes are continuous as well [Theorem 4.24].

Theorem 4.23. *Suppose we have m dimensional integral current spaces, $M_i = (X_i, d_i, T_i)$, such that $M_i \xrightarrow{\mathcal{F}} M_\infty$, then for any fixed $\epsilon > 0$, their interval filling volumes converge,*

$$(354) \quad \lim_{i \rightarrow \infty} \mathbb{IFV}_\epsilon(M_i) = \mathbb{IFV}_\epsilon(M_\infty)$$

.

Proof. By Proposition 3.9, we see that $M_i \times I_\epsilon \xrightarrow{\mathcal{F}} M_\infty \times I_\epsilon$. Thus we have continuity applying Theorem 2.46. \square

We now prove the continuity of the sliced interval filling volume defined in Definition 3.45.

Theorem 4.24. *Suppose $M_i \xrightarrow{\mathcal{F}} M$ and $\text{Diam}(M_i) \leq D_0$. If $p_i \in M_i$ converge to $p_\infty \in M_\infty$, and $q_{j,i} \in M_\infty$ converge to $q_{j,\infty} \in M_\infty$ for $j = 1..k$ then for any fixed $\epsilon > 0$,*

$$(355) \quad \lim_{i \rightarrow \infty} \mathbb{SIF}_\epsilon(p_i, r, q_{1,i}, \dots, q_{k,i}) = \mathbb{SIF}_\epsilon(p_\infty, r, q_{1,\infty}, \dots, q_{k,\infty}).$$

Proof. This theorem is a consequence of Proposition 4.25 stated and proven immediately below, combined with Proposition 4.18 and Definition 3.45. \square

Proposition 4.25. *Suppose we have a sequence of m dimensional integral current spaces, $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i : X_i \rightarrow Z$, such that*

$$(356) \quad \lim_{i \rightarrow \infty} d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0$$

and points $z_{j,i} \in Z$ such that $\delta_{j,i} = d_Z(z_{j,i}, z_{j,\infty})$ for $j = 1..k$ for some $k \in \{0, \dots, m-1\}$ and $p_i \in X_i$, such that $\delta_i = d_Z(\varphi_i(p_i), \varphi(p))$, then for any fixed $\epsilon > 0$ and almost every $t \in \mathbb{R}^k$

$$(357) \quad |\mathbb{IFV}_\epsilon(\text{Slice}(M_i, \rho_i, r)) - \mathbb{IFV}_\epsilon(\text{Slice}(M_\infty, \rho_\infty, r))| \leq (2 + \epsilon) \left(D_i + \sum_{j=1}^k (a_j + b_j) \right)$$

where a_j, b_j are defined in (332)-(333) and ρ_i is defined in (335). If $\lim_{i \rightarrow \infty} \delta_{j,i} = 0$ then for almost every $r \in \mathbb{R}$ and $r \in \mathbb{R}^k$, the filling volumes of slices converge,

$$(358) \quad \lim_{i \rightarrow \infty} \mathbb{IFV}_\epsilon(\text{Slice}(M_i, \rho_i, t)) = \mathbb{IFV}_\epsilon(\text{Slice}(M_\infty, \rho_\infty, t)).$$

Furthermore, if $\text{Diam}(M_i) \leq D_0$ then

$$(359) \quad \lim_{i \rightarrow \infty} \int_{t \in [-R, R]^k} \mathbb{IFV}_\epsilon(\text{Slice}(M_i, \rho_i, t)) \mathcal{L}^k = \int_{t \in [-R, R]^k} \mathbb{IFV}_\epsilon(\text{Slice}(M_\infty, \rho_\infty, t)) \mathcal{L}^k.$$

where

$$(360) \quad R = 2D_0 + \sup\{\delta_{ji} : i = 1..\infty, j = 1..k\} < \infty\}$$

Proof. Let $D_i = d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty)$ so $D_i \rightarrow 0$. By Proposition 4.19 we have

$$(361) \quad d_{\mathcal{F}}(\text{Slice}(M_i, \rho_i, r), \text{Slice}(M_\infty, \rho_\infty, r)) \leq D_i + \sum_{j=1}^k (a_j + b_j).$$

Next applying Proposition 3.9, we have

$$(362) \quad d_{\mathcal{F}}(\text{Slice}(M_i, \rho_i, r) \times I_\epsilon, \text{Slice}(M_\infty, \rho_\infty, r) \times I_\epsilon) \leq (2 + \epsilon) \left(D_i + \sum_{j=1}^k (a_j + b_j) \right).$$

By Theorem 2.46 we then have

$$(363) \quad |\text{FillVol}(\partial N_i) - \text{FillVol}(\partial N_\infty)| \leq d_{\mathcal{F}}(N_1, N_2).$$

Taking $N_i = \text{Slice}(M_i, \rho_i, r) \times I_\epsilon$, we have (357). which gives us (358). We cannot just integrate and obtain a limit as in (359) because a_j and b_j are functions. So we observe that

$$\begin{aligned} \int_{[-R,R]^k} \mathbb{IFV}_\epsilon(\text{Slice}(M_\infty, \rho_i, t)) \mathcal{L}^k &\leq \int_{[-R,R]^k} \mathbf{M}(\text{Slice}(M_\infty, \rho_i, t)) \mathcal{L}^k \\ &\leq \mathbf{M}(M_\infty) \\ \int_{[-R,R]^k} a_j(t) \mathcal{L}^k &\leq \int_{[-R,R]^k} \mathbf{M}(\text{Slice}(T_\infty, \rho_\infty, t)) \mathcal{L}^k \\ &\leq \mathbf{M}(M_\infty) \\ \int_{[-R,R]^k} b_j(t) \mathcal{L}^k &\leq \int_{[-R,R]^k} \mathbf{M}(\text{Slice}(\partial T_\infty, \rho_\infty, t)) \mathcal{L}^k \\ &\leq \mathbf{M}(\partial M_\infty). \end{aligned}$$

Thus we may apply the Dominated Convergence Theorem of Real Analysis to obtain the limit of the integrals in (359). \square

4.6. Existence of Limits of Points. Recall that given an integral current space (X, d, T) , $X = \text{set}(T)$ the collection of points of positive density and the metric completion $\bar{X} = \text{spt}(T)$, and any isometric embedding $\varphi : X \rightarrow Z$ maps X isometrically onto $\text{set}(\varphi_\#T)$ and extends to $\varphi : \bar{X} \rightarrow Z$ which maps isometrically onto $\text{spt}(\varphi_\#T)$. A point $z \in \text{spt}(S)$ iff for every $r > 0$, $T \llcorner \bar{B}(p, r) \neq 0$.

Recall Definition 3.22 of $\mathbf{SF}_k(p, r)$ and that $\mathbf{SF}_0(p, r) = \text{FillVol}(\partial S(p, r))$.

Proposition 4.26. *If M_i are integral current spaces and $d_{\mathcal{F}}(M_i, M_\infty) \rightarrow 0$ and $p \in \bar{M}_\infty$, then there exists $p_i \in M_i$ and $r_p > 0$ such that p_i converge to p and for any such converging sequence of p_i ,*

$$(364) \quad \mathbf{M}(S(p_i, r)) \geq \mathbf{SF}_k(p_i, r) \geq C_k > 0$$

for almost every $r < r_p$.

If $p_\infty \in M_\infty$, then there exists $p_i \in M_i$ and $r_p > 0$ and $C_p > 0$ such that p_i converge to p and any such converging sequence has $\mathbf{M}(S(p_i, r)) > C_p r^k$ for all $r < r_p$.

Proof. If $p \in \bar{M}_\infty$, then for all $r > 0$, $B_p(r) \cap M \neq \emptyset$. So for almost every $r > 0$, $S(p, r) \neq 0$ and $\mathbf{SF}_k(p, r) \neq 0$. Taking $z_\infty = \varphi_\infty(p)$, we know $\varphi_{\infty\#}T_\infty \cap B(z_\infty, r)$ is a nonzero integral current and since $\varphi_{i\#}T_i$ converge to $\varphi_{\infty\#}T_\infty$ in Z , we know that for almost every $r > 0$,

there exists N_r sufficiently large that $S_i = \varphi_{i\#}T_i \cap B(z_\infty, r)$ is a nonzero integral current. In particular there is a point

$$(365) \quad z_i \in \text{set}(\varphi_{i\#}T_i) \cap B(z_\infty, r).$$

Since $\varphi_i : X_i \rightarrow \text{set}(\varphi_{i\#}T_i)$ is an isometry, there exists $p_i \in X_i$, such that $\varphi_i(p_i) = z_i \rightarrow z_\infty$.

By Theorem 4.20, we know

$$(366) \quad \lim_{i \rightarrow \infty} \mathbf{SF}_k(p_i, r) = \mathbf{SF}_k(p, r)$$

and

$$(367) \quad \liminf_{i \rightarrow \infty} \mathbf{M}(S(p_i, r)) \geq \mathbf{M}(S(p, r))$$

which is nonzero when $p \in \bar{M}_\infty$ and is $\geq Cr^m$ when $p = p_\infty \in M_\infty$. \square

Theorem 4.27. *Suppose M_i are integral current spaces of dimension m with M_i converge to M_∞ in the intrinsic flat sense and $p_i \in M_i$ are Cauchy. If there is a uniform constant $C > 0$ such that for some $k \in \{0, 1, \dots, m-1\}$ we have*

$$(368) \quad S(p_i, r) \neq 0 \text{ and } \mathbf{SF}_k(B(p_i, r)) \geq C,$$

then p_i converge to a point $p_\infty \in \bar{M}_\infty$. If we have

$$(369) \quad \mathbf{SF}_k(B(p_i, r)) \geq C_{SF} r^m > 0,$$

then p_i converge to a point $p_\infty \in M_\infty$.

Proof. By the definition of a Cauchy sequences of points $p_i \in M_i$ we have the hypothesis of Theorem 4.20 with $z_i = \varphi_i(p_i)$ and a point $z_\infty = \lim_{i \rightarrow \infty} z_i$.

Thus applying Theorem 4.20 to balls $S(z_i, r)$ which converge to $S(z, r)$ for almost every $r > 0$ as in Proposition 4.18, we have

$$(370) \quad \int_{t \in [-R, R]^k} \text{FillVol}(\partial \text{Slice}(S(z_\infty, r), \rho_\infty, t)) \mathcal{L}^k \geq C > 0$$

By Lemma 3.21 we have

$$(371) \quad \mathbf{M}(\varphi_{\infty\#}T_\infty \llcorner \bar{B}(z, r)) = \mathbf{M}(S(z_\infty, r)) \geq C > 0$$

and so $\varphi_{i\#}T_i \cap B(z_\infty, r) \neq 0$. So $z_\infty \in \text{spt}(T_\infty) = Cl(\text{set}(T_\infty))$. So there exists $p_\infty \in \bar{X}_\infty$ such that $\varphi_\infty(p_\infty) = z_\infty$. Thus p_i have a limit point $p_\infty \in \bar{M}_\infty$.

Once again applying Theorem 4.20 to balls $S(z_i, r)$ which converge to $S(z, r)$ as in Proposition 4.18, we have

$$(372) \quad \int_{t \in [-R, R]^k} \text{FillVol}(\partial \text{Slice}(S(z_\infty, r), \rho_\infty, t)) \mathcal{L}^k \geq C_{SF} r^m.$$

By Lemma 3.21 we have

$$(373) \quad \liminf_{r \rightarrow 0} \frac{\mathbf{M}(\varphi_{\infty\#}T_\infty \llcorner \bar{B}(z, r))}{r^m} = \liminf_{r \rightarrow 0} \frac{\mathbf{M}(S(z_\infty, r))}{r^m} \geq C_{SF} > 0$$

and so $z_\infty \in \text{set}(\varphi_{\infty\#}T_\infty)$, and so there exists $p_\infty \in X_\infty$ such that $\varphi_\infty(p_\infty) = z_\infty$. Thus p_i have a limit point $p_\infty \in M_\infty$. \square

Example 4.28. *It is quite possible for a Cauchy sequence of points to have more than one limit as can be seen simply by taking the constant sequence of integral current spaces, S^1 , and noting that due to the isometries, any point may be set up as the limit of a Cauchy sequence of points. One may also use isometries of S^1 to relocate a Cauchy sequence so that the images are no longer Cauchy in Z .*

I believe I can prove uniqueness up to isometry using techniques from [18].

4.7. Bolzano-Weierstrass Theorem. I need to tex up a Bolzano-Weierstrass Theorem which states that any sequence $p_i \in M_i$ where M_i converge in the intrinsic flat sense to a compact integral current space M_∞ where we assume lower filling estimates around p_i have a subsequence converging to a point $p_\infty \in M_\infty$. I believe I can remove the compactness condition by assuming strong enough lower filling estimates so I have not texed this up yet.

4.8. Arzela-Ascoli Theorems.

Theorem 4.29. *Suppose $M_i = (X_i, d_i, T_i)$ are integral current spaces and M_i converge in the intrinsic flat sense to M_∞ and $F_i : X_i \rightarrow W$ are Lipschitz maps into a compact metric space W with $\text{Lip}(F_i) \leq K$, then a subsequence converges to a Lipschitz map $F_\infty : X_\infty \rightarrow W$ with $\text{Lip}(F_\infty) \leq K$. More specifically, there exists isometric embeddings $\varphi_i : X_i \rightarrow Z$ such that $d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) \rightarrow 0$ and for any sequence $p_i \in X_i$ converging to $p \in X_\infty$ (i.e. $d_Z(\varphi_i(p_i), \varphi_\infty(p)) \rightarrow 0$, we have $\lim_{i \rightarrow \infty} F_i(p_i) = F_\infty(p)$).*

Note that the corresponding Gromov-Hausdorff Arzela-Ascoli Theorem allows the target spaces to vary as long as they converge in the Gromov-Hausdorff sense. See for example Grove-Petersen [11]. Here we cannot allow the target space to vary and converge only in the intrinsic flat sense even if the domain is fixed as can be seen in examples where the intrinsic flat limits are no longer connected.

Proof. By Theorem 2.40 we have $\varphi_i : M_i \rightarrow Z$ such that $d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) \rightarrow 0$. Take any $p \in X_\infty$. By Theorem FILLIN, there exists $p_i \in X_i$ such that $\lim_{i \rightarrow \infty} \varphi_i(p_i) = \varphi_\infty(p)$. Their images $F_i(p_i) \in W$ have a subsequence which converges to some $w \in W$. We set $F_\infty(p) = w$. Recall that integral current spaces are separable. So we have a countable dense subset $X_0 \subset X_\infty$. So we may repeat this process creating subsequences of subsequences for a countable dense collection of $p \in X_0 = X_\infty$. Diagonalizing, we have defined a function $F_\infty : X_0 \subset X_\infty \rightarrow K$.

Observe that for all $p, q \in X_0$ we have p_i and q_i converging to them such that

$$(374) \quad d_W(F_\infty(p), F_\infty(q)) = \lim_{i \rightarrow \infty} d_W(F_i(p_i), F_i(q_i))$$

$$(375) \quad \leq \lim_{i \rightarrow \infty} K d_{X_i}(p_i, q_i)$$

$$(376) \quad \leq \lim_{i \rightarrow \infty} K d_Z(\varphi_i(p_i), \varphi_i(q_i))$$

$$(377) \quad \leq K d_Z(\varphi_\infty(p), \varphi_\infty(q))$$

$$(378) \quad \leq K d_{X_\infty}(p, q).$$

Thus we may extend F_∞ continuously to $F_\infty : X_\infty \rightarrow W$ and $\text{Lip}(F_\infty) \leq K$.

Now suppose we have an arbitrary sequence $p_i \in X_i$ such that $d_Z(\varphi_i(p_i), \varphi_\infty(p)) \rightarrow 0$ then there exists $q_j \in X_0$ converging to $p_\infty \in X$ with

$$(379) \quad d_W(F_\infty(p_\infty), F_\infty(q_j)) \leq K d_{X_\infty}(p_\infty, q_j)$$

By the definition of F_∞ we have have $q_{j,i} \in X_i$ with $d_Z(\varphi_i(q_{j,i}), \varphi_\infty(q_j)) \rightarrow 0$ and

$$(380) \quad \lim_{i \rightarrow \infty} d_W(F_i(q_{j,i}), F_\infty(q_j)) = 0.$$

Observe that

$$\begin{aligned}
 (381) \quad \lim_{i \rightarrow \infty} d_W(F_i(p_i), F_i(q_{j,i})) &\leq \lim_{i \rightarrow \infty} K d_{X_i}(p_i, q_{j,i}) \\
 (382) \quad &\leq \lim_{i \rightarrow \infty} K d_Z(\varphi_i(p_i), \varphi_i(q_{j,i})) \\
 (383) \quad &\leq K d_Z(\varphi_\infty(p_\infty), \varphi_\infty(q_j)) \\
 (384) \quad &= K d_{X_\infty}(p_\infty, q_j)
 \end{aligned}$$

Combining these we have

$$\begin{aligned}
 (385) \quad \lim_{i \rightarrow \infty} d_W(F_i(p_i), F_\infty(p_\infty)) &\leq \lim_{i \rightarrow \infty} d_W(F_i(p_i), F_i(q_{j,i})) \\
 (386) \quad &+ \lim_{i \rightarrow \infty} d_W(F_i(q_{j,i}), F_\infty(q_j)) + d_W(F_\infty(p_\infty), F_\infty(q_j)) \\
 (387) \quad &\leq K 2 d_Z(\varphi_\infty(p_\infty), \varphi_\infty(q_j))
 \end{aligned}$$

Taking $j \rightarrow \infty$ we have our claim. \square

I believe I can prove an Arzela-Ascoli Theorem for uniformly bi-Lipschitz maps between M_i and N_i where M_i and N_i converge in the Intrinsic Flat sense. I definitely can prove isometries converge to isometries but have not texed this as I work on this slightly harder theorem.

4.9. Compactness Theorems Improved.

Theorem 4.30. *Given a sequence of m dimensional integral current spaces $M_i = (X_i, d_i, T_i)$ with $\mathbf{M}(M_i) \leq V_0$, $\mathbf{M}(\partial M_i) \leq A_0$, $\text{Diam}(M_i) \leq D_0$ and a uniform constant $C_{SF} > 0$ such that for some $k \in 0..(m-1)$ we have*

$$(388) \quad \mathbf{SF}_k(p, r) \geq C_{SF} r^m$$

then a subsequence of the M_i converge in the Gromov-Hausdorff sense and the Intrinsic Flat sense to a nonzero integral current space M_∞ .

Proof. By Theorem 3.23, we know a subsequence (X_i, d_i) has a Gromov-Hausdorff limit (Y, d_Y) . Thus by Gromov, there exists a common compact metric space Z and isometric embeddings $\varphi_i : X_i \rightarrow Z$, $\varphi : Y \rightarrow Z$, such that $d_H^Z(\varphi(X_i), \varphi(Y)) \rightarrow 0$. By Ambrosio-Kirchheim Compactness Theorem, a subsequence of $\varphi_{i\#} T_i$ converges to $T_\infty \in \mathbf{I}_m(Z)$. Let $M_\infty = (\text{set}(T_\infty), d_Z, T_\infty)$.

We need only show $\varphi(Y) = \text{set}(T_\infty)$. Let $z_\infty \in Y$, and let $p_i \in X_i$ such that $z_i = \varphi_i(p_i) \rightarrow z$. By Theorem 4.27, we see that $z_\infty = \varphi(p_\infty)$. \square

Theorem 4.31. *Given $r_0 > 0, \beta \in (0, 1), C > 0, V_0 > 0, A_0 > 0$. If a sequence of compact Riemannian manifolds, M^m , has $\text{Vol}(M^m) \leq V_0$, $\text{Diam}(M^m) \leq D_0$ and the C, β (integral) tetrahedral property for all balls of radius $\leq r_0$, then a subsequence converges in the Gromov-Hausdorff and Intrinsic Flat sense to a nonzero integral current space.*

Here our manifolds do not have boundary.

Proof. The C, β (integral) tetrahedral property implies that there exists $C_{SF} > 0$ such that

$$(389) \quad \mathbf{SF}_{m-1}(p, r) \geq C_{SF} r^m.$$

Theorem 3.41 implies there exists a uniform upper bound on diameter. So we apply Theorem 4.30. \square

Insert a corresponding compactness theorem for $k = m - 2$.

Remark 4.32. *As a consequence of this theorem, we see that there is no uniform tetrahedral property on manifolds with positive scalar curvature even when the volume of the balls are uniformly bounded below by that of Euclidean balls. In fact there exist a sequence of such manifolds, M_j^3 , whose intrinsic flat limit is 0 described in [19].*

4.10. Consequences of Continuity. Here we will itemize the consequence of the continuity of \mathbf{SF}_k and $\text{FillVol}(S(p, r))$ on classes of spaces where intrinsic flat and Gromov-Hausdorff limits of spaces agree including the classes in our compactness theorems given above and manifolds with nonnegative Ricci curvature and uniform lower bounds on volume.

There are also some nice consequences of knowing a sequence converges to a ball in Euclidean space or to a sphere or to a manifold that I will put here.

4.11. Slicing to Specific Limits. In order to prove conjectures like those stated in [13] and [14], one needs to prove a sequence of Riemannian manifolds has a specific intrinsic flat limit. In this section we will prove theorems which allow one to prove results about slices in order to achieve a specific limit.

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